

# The three-loop form factor in $\mathcal{N} = 4$ super Yang-Mills

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**Thomas Gehrmann<sup>a,b</sup>, Johannes M. Henn<sup>b,c,d</sup>, Tobias Huber<sup>b,e</sup>**

<sup>a</sup> *Institut für Theoretische Physik, Universität Zürich,  
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland*

<sup>b</sup> *Kavli Institute for Theoretical Physics  
University of California, Santa Barbara, CA 93106, USA*

<sup>c</sup> *Institut für Physik, Humboldt-Universität zu Berlin,  
Newtonstraße 15, D-12489 Berlin, Germany*

<sup>d</sup> *Institute for Advanced Study, Princeton, NJ 08540, USA*

<sup>e</sup> *Theoretische Physik 1, Naturwissenschaftlich-Technische Fakultät,  
Universität Siegen, Walter-Flex-Strasse 3, D-57068 Siegen, Germany*

thomas.gehrmann@uzh.ch,  
jmhenn@ias.edu,  
huber@tp1.physik.uni-siegen.de

**ABSTRACT:** In this paper we study the Sudakov form factor in  $\mathcal{N} = 4$  super Yang-Mills theory to the three-loop order. The latter is expressed in terms of planar and non-planar loop integrals. We show that it is possible to choose a representation in which each loop integral has uniform transcendentality. We verify analytically the expected exponentiation of the infrared divergences with the correct values of the three-loop cusp and collinear anomalous dimensions in dimensional regularisation. We find that the form factor in  $\mathcal{N} = 4$  super Yang-Mills can be related to the leading transcendentality part of the quark and gluon form factors in QCD. We also study the ultraviolet properties of the form factor in  $D > 4$  dimensions, and find unexpected cancellations, resulting in an improved ultraviolet behaviour.

**KEYWORDS:** Supersymmetric gauge theory, NLO Computations.

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## 1. Introduction

In this paper we study the Sudakov form factor in  $\mathcal{N} = 4$  super Yang-Mills (SYM) with gauge group  $SU(N)$ . Following van Neerven [1], we study the vacuum expectation value of an operator built from two scalars, inserted into two on-shell states. The operator belongs to the stress-energy supermultiplet, which contains the conserved currents of  $\mathcal{N} = 4$  SYM, and has zero anomalous dimension. Together with the vanishing  $\beta$  function of  $\mathcal{N} = 4$  SYM this means that the form factor is ultraviolet (UV) finite in four dimensions. Therefore only infrared (IR) divergences associated to the on-shell states appear, which we regularise using dimensional regularisation.

Generalisations of the Sudakov form factor to the case of different composite operators, and more external on-shell legs, have been discussed recently in refs. [2–5]. Form factors have also been studied within the AdS/CFT correspondence in the dual AdS description, see refs. [6, 7]. Here we will focus on the perturbative expansion of the form factor of ref. [1].

Form factors are closely related to scattering amplitudes. For example, planar amplitudes can be factorised into an infrared divergent part, given by a product of form factors, and an infrared finite remainder (a ‘hard’ function in QCD terminology), see e.g. ref. [8] and references therein. The infrared divergent part exponentiates and has a simple universal form. In fact, for four- and five-point scattering amplitudes the exponentiation property of the divergent part carries over to the finite part as well [8, 9]. This is a consequence of a hidden dual conformal symmetry of planar scattering amplitudes. The latter relates the finite part to the infrared divergent part through a Ward identity [10, 11]. The relation to form factors makes it possible to give an operator definition of the finite remainder. The scheme independence of the latter was recently checked in a two-loop computation using dimensional and massive regularisations [12].

Scattering amplitudes in  $\mathcal{N} = 4$  SYM have many special properties, and it is interesting to ask how much of this simplicity carries over to the form factors. For both the planar four-particle amplitude and the form factor, the general form of the result is known in principle. For the former, this is due to dual conformal symmetry, and for the latter it is due to the exponentiation of infrared divergences. However it is quite non-trivial to obtain these a priori known results from explicit perturbative calculations, evaluating loop integrals. The simplicity of the final results suggests that there should be more structure hidden in the loop integral expressions, and by studying them further one might gain insights into better ways of evaluating them, which is of more general interest.

One might expect that the evaluation of form factors should be simpler than that of scattering amplitudes, as the former have a trivial scale dependence only, whereas the latter are functions of ratios of Mandelstam variables, e.g.  $s/t$  in the four-point case. Given this, it is somewhat surprising that less is known about the loop expansion of form factors in  $\mathcal{N} = 4$  SYM than about scattering amplitudes. For example, while the planar four-point amplitude was evaluated to the four-loop order (in part numerically) [13–15], the form factor has only been computed to the two-loop order in ref. [1], in a calculation that dates back to 1986. In the present paper, we extend the calculation of ref. [1] to three loops, and study which of the properties that have been observed for scattering amplitudes are present.

One fact which makes form factors technically challenging compared to planar amplitudes, however also more interesting, is the following. At leading order in the ‘t Hooft limit  $N \rightarrow \infty$ , where the coupling  $\lambda = g^2 N$  is kept fixed, both planar as well as non-planar integrals appear in the form factor. This is easily understood by the fact that the operator insertion is a colour-singlet. It is interesting to note that the non-planar diagrams appearing in the form factor are related, through the unitarity technique, to a priori subleading double trace terms in the four-particle scattering amplitude. Therefore, the form factor at leading order in  $N$  contains information about non-planar corrections to the four-particle amplitude. The first non-planar diagram, the crossed ladder, appears at the two-loop level. At three loops, we find five different non-planar diagrams that contribute, i.e. that have non-vanishing coefficient.

It is an observed, albeit unproven fact that results for scattering amplitudes in  $\mathcal{N} = 4$  super Yang-Mills have uniform transcendentality (UT), i.e. can be expressed as linear

combinations of polylogarithmic functions of uniform degree  $2L$ , where  $L$  is the loop order, with constant coefficients. In  $\epsilon$ -expansions of dimensionally regularised quantities which depend only on a single scale, the coefficients of the Laurent expansion in  $\epsilon$  are real constants which are in general of increasing transcendentality in the Riemann  $\zeta$ -function. In this context uniform transcendentality refers to homogeneity in the degree of transcendentality ( $DT$ ), where the latter is defined as

$$\begin{aligned} DT(r) &= 0 \quad \text{for rational } r \\ DT(\pi^k) &= DT(\zeta_k) = k \\ DT(x \cdot y) &= DT(x) + DT(y) . \end{aligned}$$

In the planar case, the property of UT is even true for individual loop integrals, at least when they are expressed in an appropriate basis of dual conformal integrals [16, 17]. Incidentally, this also has practical advantages, as these integrals are easier to evaluate [17, 18] than those in other representations. Dual conformal symmetry is only expected in the planar case, but what can be said about the transcendentality properties of non-planar integrals? At four points, the non-planar double ladder integral is not of uniform transcendentality. However, if defined with an appropriate loop-dependent numerator factor, it does have this property [19, 20]. Changing to a basis involving the latter integral allows one to understand the UT property of four-point non-planar  $\mathcal{N} = 4$  SYM amplitudes [21] and  $\mathcal{N} = 8$  supergravity amplitudes [22, 23]. It also raises the interesting question whether this is a generic feature.

All planar and non-planar master integrals for form factors in dimensional regularisation at three loops are known from the computation of the form factor in QCD [24–33], and some of them have UT, while others do not. It has been observed that some of the integrals do have UT if they are defined with certain (loop-dependent) numerator factors [19]. The latter resemble the numerator factors required by dual conformal symmetry in the planar case [34]. In this paper, we find similar numerator factors for all topologies with 7, 8, 9 propagators, such that the integrals have UT. Moreover, we find that the complete three-loop form factor can be written solely in terms of UT integrals.

Finding a representation that has this property required using certain identities for non-planar form factor integrals that are based on reparametrisation invariances, which we found as a by-product of our analysis. They generalise an identity found by Davydychev and Usyukina [35].

As was already mentioned, in  $\mathcal{N} = 4$  SYM, scattering amplitudes and the form factors studied here are UV finite in four dimensions. It is interesting to ask in what dimension, called critical dimension  $D_c$ , they first develop UV divergences. This question is of theoretical interest in the context of the discussion of possible finiteness of  $\mathcal{N} = 8$  supergravity, see e.g. [36–38] and references therein. More practically, bounds on the critical UV dimension at a given loop order can also be a useful cross-check of computations, or constrain the types of loop integrals that can appear. Ultraviolet power counting, based on the existence of  $\mathcal{N} = 3$  off-shell superspace [39], provides a lower bound for the critical dimension. We analyse the UV properties of the form factor to three loops and find that at each loop

order, the critical dimension is  $D_c = 6$ . This is consistent with the bound obtained from superspace power counting. We find that the latter bound is saturated at two loops, while it is too conservative at three loops, where the ultraviolet behaviour is better than suggested by the bound. This is the result of a cancellation between different loop integrals. We find a representation where the UV behaviour is manifest.

This paper is organised as follows. We review the known expression for the form factor to two loops in section 2. We then discuss identities for non-planar integrals to three loops in section 3. In section 4, using the unitarity-based method, we derive an expression for the three-loop form factor in terms of loop integrals. We then evaluate the latter in section 5 and verify the exponentiation of infrared divergences in section 6. We then analyse the ultraviolet properties of the form factor to three loops in section 7. We conclude in section 8. There are several appendices. Appendix A contains the analytic expressions of the  $\epsilon$  expansion of the integrals used in the paper, while appendix B contains the expression of the form factor in terms of conventionally used master integrals. Finally, appendix C reviews the on-shell four-point amplitude to two loops that is used in the unitarity calculation in the main text.

## 2. Form factor to two loops

In order to define the scalar form factor in  $\mathcal{N} = 4$  SYM, we start by introducing the bilinear operator

$$\mathcal{O} = \text{Tr}(\phi_{12}\phi_{12}), \quad (2.1)$$

where the scalars  $\phi_{AB}$  are in the representation **6** of  $SU(4)$ , and  $\phi_{AB} = \phi_{AB}^a T_a$ , with  $T_a$  being the generators of  $SU(N)$  in the fundamental representation, normalised according to  $\text{Tr}(T^a T^b) = \delta^{ab}$ . This operator is a particular component of the stress-energy supermultiplet of  $\mathcal{N} = 4$  SYM, and has zero anomalous dimension. We then define the form factor as the vacuum expectation value of  $\mathcal{O}$  inserted into two on-shell states in the adjoint representation,

$$\mathcal{F}_S = \langle \phi_{34}^a(p_1) \phi_{34}^b(p_2) \mathcal{O} \rangle, \quad (2.2)$$

with the convention that momentum is outgoing.

Since  $\mathcal{O}$  is a colour singlet, the form factor must be proportional to  $\text{Tr}(T^a T^b)$ ,

$$\mathcal{F}_S = \text{Tr}(T^a T^b) F_S. \quad (2.3)$$

We work in dimensional regularisation with  $D = 4 - 2\epsilon$  dimensions in order to regulate IR divergences associated with the on-shell legs. We write the form factor as an expansion in the 't Hooft coupling [8]

$$a = \frac{g^2 N}{8\pi^2} (4\pi)^\epsilon e^{-\epsilon\gamma_E}, \quad (2.4)$$

according to

$$F_S = 1 + a x^\epsilon F_S^{(1)} + a^2 x^{2\epsilon} F_S^{(2)} + a^3 x^{3\epsilon} F_S^{(3)} + \mathcal{O}(a^4). \quad (2.5)$$

We normalized the tree-level contribution to unity and introduced

$$x = \frac{\mu^2}{-q^2 - i\eta} , \quad (2.6)$$

with the infinitesimal quantity  $\eta > 0$ .

We remark that the dependence on the number of colours  $N$  in equation (2.5) is exact. In order to see this, let us show that the three-loop contribution to the form factor must be proportional to  $N^3$  (a similar analysis trivially holds at one and two loops).

The reasoning is as follows. Imagine a generic Feynman diagram contributing to  $\mathcal{F}_S$ . Without loss of generality, suppose that it is built from three-point vertices, whose colour dependence is given by the structure constants  $f^{a_1 a_2 a_3}$ . For each internal line, there is a sum over adjoint colour indices, with the result being proportional to  $\text{Tr}(T^a T^b)$ , as stated in equation (2.3). Our goal is to determine the proportionality factor. In order to do this, it is convenient to sum also over the free indices  $a$  and  $b$ ,

$$\sum_{a,b} \delta^{ab} \text{Tr}(T^a T^b) = N^2 - 1 . \quad (2.7)$$

We can then represent each Feynman diagram as a circle with inscribed lines. There are three inequivalent structures that can appear,

$$\begin{aligned} A &= f^{abg} f^{bcg} f^{cdh} f^{edi} f^{efi} f^{fah} , \\ B &= f^{abg} f^{bch} f^{cdg} f^{dei} f^{eif} f^{fha} , \\ C &= f^{abg} f^{bch} f^{cdi} f^{deg} f^{ehf} f^{fia} , \end{aligned} \quad (2.8)$$

which correspond to the case of zero, one, or two intersections of the inscribed lines, respectively. Sums over repeated indices are implicit. In order to carry out the sums, it is convenient to write the structure constants as

$$f^{abc} = -i/\sqrt{2} \left( \text{Tr}[T^a T^b T^c] - \text{Tr}[T^a T^c T^b] \right) . \quad (2.9)$$

Using the  $SU(N)$  Fierz identities,

$$\sum_a \text{Tr}(AT^a) \text{Tr}(BT^a) = \text{Tr}(AB) - 1/N \text{Tr}(A) \text{Tr}(B) , \quad (2.10)$$

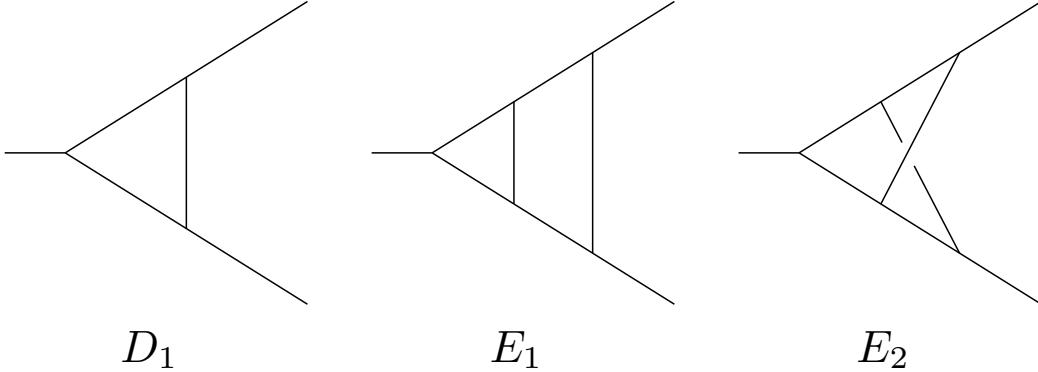
$$\sum_a \text{Tr}(AT^a BT^a) = \text{Tr}(A) \text{Tr}(B) - 1/N \text{Tr}(AB) , \quad (2.11)$$

one easily finds

$$A = (N^2 - 1)N^3 , \quad B = -\frac{1}{2}(N^2 - 1)N^3 , \quad C = 0 . \quad (2.12)$$

Taking into account equation (2.7), we see that  $\mathcal{F}_S$  at three loops is proportional to  $N^3$ , as claimed.

Note that beginning from four loops there can be more than one colour structure, and in particular the quartic Casimir can appear. An explicit example of this is the four-loop



**Figure 1:** Diagrams that contribute to the one-loop and two-loop form factors  $\mathcal{F}_S^{(1)}$  and  $\mathcal{F}_S^{(2)}$  in  $\mathcal{N} = 4$  SYM. All internal lines are massless. The incoming momentum is  $q = p_1 + p_2$ , outgoing lines are massless and on-shell, i. e.  $p_1^2 = p_2^2 = 0$ . All diagrams displayed have unit numerator and exhibit uniform transcendentality (UT) in their Laurent expansion in  $\epsilon = (4 - D)/2$ .

contribution to the QCD  $\beta$  function [40]. An interesting related question has to do with the colour dependence of infrared divergences in gauge theories, see e.g. [41], and references therein.

The form factor to two loops was computed a long time ago [1]. It contains as building blocks the diagrams displayed in Fig. 1 and reads

$$\begin{aligned} F_S &= 1 + g^2 N \mu^{2\epsilon} \cdot (-q^2) \cdot 2 D_1 + g^4 N^2 \mu^{4\epsilon} \cdot (-q^2)^2 \cdot [4 E_1 + E_2] + \mathcal{O}(g^6) \\ &= 1 + a x^\epsilon R_\epsilon \cdot 2 D_1^{\text{exp}} + a^2 x^{2\epsilon} R_\epsilon^2 \cdot [4 E_1^{\text{exp}} + E_2^{\text{exp}}] + \mathcal{O}(a^3) , \end{aligned} \quad (2.13)$$

with

$$R_\epsilon \equiv \frac{e^{\epsilon \gamma_E}}{2 \Gamma(1 - \epsilon)} . \quad (2.14)$$

The expressions for  $D_1$ ,  $D_1^{\text{exp}}$ ,  $E_i$ , and  $E_i^{\text{exp}}$  are given explicitly in appendix A and result in

$$\begin{aligned} F_S^{(1)} &= R_\epsilon \cdot 2 D_1^{\text{exp}} \\ &= -\frac{1}{\epsilon^2} + \frac{\pi^2}{12} + \frac{7\zeta_3}{3} \epsilon + \frac{47\pi^4}{1440} \epsilon^2 + \epsilon^3 \left( \frac{31\zeta_5}{5} - \frac{7\pi^2\zeta_3}{36} \right) + \epsilon^4 \left( \frac{949\pi^6}{120960} - \frac{49\zeta_3^2}{18} \right) \\ &\quad + \epsilon^5 \left( -\frac{329\pi^4\zeta_3}{4320} - \frac{31\pi^2\zeta_5}{60} + \frac{127\zeta_7}{7} \right) + \epsilon^6 \left( \frac{49\pi^2\zeta_3^2}{216} - \frac{217\zeta_3\zeta_5}{15} + \frac{18593\pi^8}{9676800} \right) \\ &\quad + \mathcal{O}(\epsilon^7) , \end{aligned} \quad (2.15)$$

$$\begin{aligned} F_S^{(2)} &= R_\epsilon^2 \cdot [4 E_1^{\text{exp}} + E_2^{\text{exp}}] \\ &= +\frac{1}{2\epsilon^4} - \frac{\pi^2}{24\epsilon^2} - \frac{25\zeta_3}{12\epsilon} - \frac{7\pi^4}{240} + \epsilon \left( \frac{23\pi^2\zeta_3}{72} + \frac{71\zeta_5}{20} \right) + \epsilon^2 \left( \frac{901\zeta_3^2}{36} + \frac{257\pi^6}{6720} \right) \\ &\quad + \epsilon^3 \left( \frac{1291\pi^4\zeta_3}{1440} - \frac{313\pi^2\zeta_5}{120} + \frac{3169\zeta_7}{14} \right) \end{aligned}$$

$$+\epsilon^4 \left(-66 \zeta_{5,3}+\frac{845 \zeta_3 \zeta_5}{6}-\frac{1547 \pi^2 \zeta_3^2}{216}+\frac{50419 \pi^8}{518400}\right)+\mathcal{O}(\epsilon^5) \ . \quad (2.16)$$

The multiple zeta values  $\zeta_{m_1, \dots, m_k}$  are defined by (see e.g. [42] and references therein)

$$\zeta_{m_1, \dots, m_k} = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \prod_{j=1}^k \frac{\text{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}. \quad (2.17)$$

The numerical values of the transcendental constants up to weight eight are:

$$\begin{aligned} \zeta_3 &= 1.2020569031595942854 \dots, & \zeta_5 &= 1.0369277551433699263 \dots, \\ \zeta_7 &= 1.0083492773819228268 \dots, & \zeta_{5,3} &= 0.037707672984847544011 \dots. \end{aligned}$$

### 3. Momentum routing invariances of integrals

Before we proceed to calculate the  $\mathcal{N} = 4$  SYM form factor to three loops via unitarity cuts, we want to investigate some of the occurring topologies more closely. In particular, we will derive identities that relate integrals without uniform transcendentality (UT) to integrals that do have this property. Since the diagrams that we will obtain from the unitarity method do not individually have UT, the following relations will be very useful later on for switching to an integral basis for the form factor in which each building block has UT.

We start with topology  $F_3^*$ , see Fig. 2. We label its incoming momentum with  $q = p_1 + p_2$ , and the outgoing ones with  $p_1$  and  $p_2$ , respectively. The latter are massless and on-shell, i.e.  $p_1^2 = p_2^2 = 0$ . The topology can be parametrised according to

$$\{k_1 - k_2, k_1 - k_3, k_1 - k_2 - k_3, k_2, k_3, k_1 - q, k_2 - q, k_3 - q, k_2 - p_1\} \ , \quad (3.1)$$

where  $k_i$  are the loop momenta. It can be seen from Fig. 2 how the momenta are distributed among the lines of the diagram  $F_3^*$ . It turns out that the following reparametrization of loop momenta,

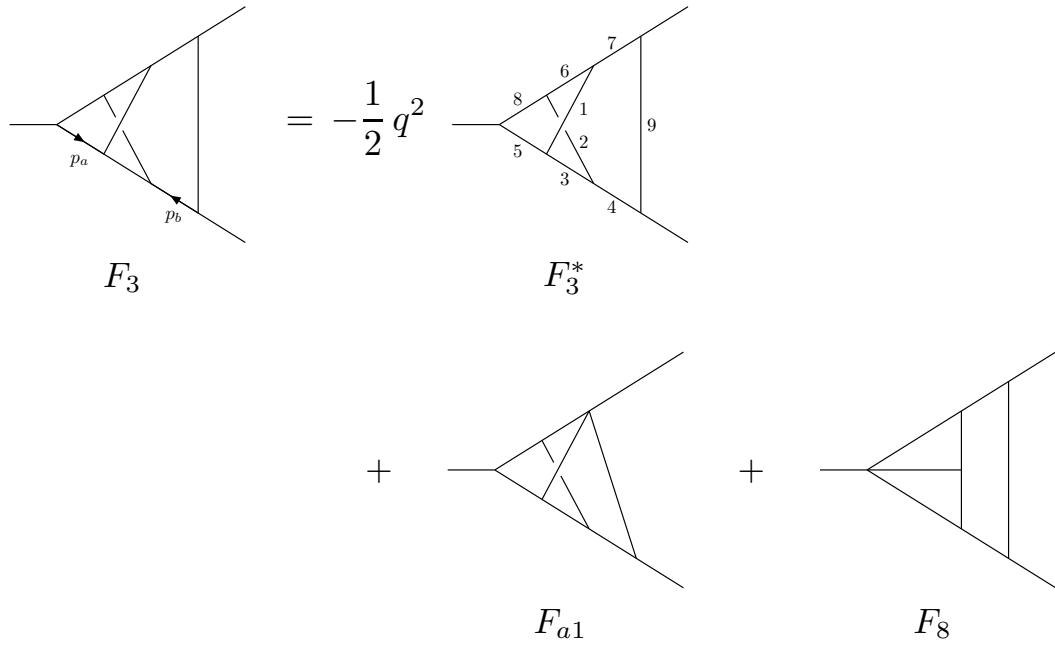
$$\begin{aligned} k_1 &\rightarrow q + k_2 - k_1 \\ k_2 &\rightarrow k_2 \\ k_3 &\rightarrow q - k_3 \ , \end{aligned}$$

does not only leave the value of the integral invariant, but even its *integrand*. We can now apply this transformation to the integral  $F_3$  which carries the factor  $(k_2 - k_3)^2$  as an irreducible scalar product in its numerator. This yields

$$\begin{aligned} (k_2 - k_3)^2 &\rightarrow (k_2 + k_3 - q)^2 \\ &= k_2^2 + k_3^2 + (k_2 - q)^2 + (k_3 - q)^2 - (k_2 - k_3)^2 - q^2 \ . \end{aligned} \quad (3.2)$$

We can now solve this equation for  $(k_2 - k_3)^2$  and get the following relation between integrals,

$$F_3 = -\frac{1}{2} q^2 F_3^* + F_{a1} + F_8 \ , \quad (3.3)$$



**Figure 2:** Diagrammatic representation of Eq. (3.3). The internal lines of all diagrams are massless. The incoming momentum is  $q = p_1 + p_2$ , outgoing lines are massless and on-shell, i.e.  $p_1^2 = p_2^2 = 0$ . Diagrams with labels  $p_a$  and  $p_b$  on arrow lines have an irreducible scalar product  $(p_a + p_b)^2$  in their numerator (diagrams that lack these labels have unit numerator). The numbers in  $F_3^*$  indicate the position of the entries in Eq. (3.1). Diagrams  $F_3$  and  $F_8$  have UT, contrary to  $F_3^*$  and  $F_{a1}$ .

which is diagrammatically shown in Fig. 2. We have now decomposed the integral  $F_3^*$ , which does not have UT in its Laurent expansion, into two integrals ( $F_3$  and  $F_8$ ) which indeed do have this property, and the auxiliary integral  $F_{a1}$ , which again does not have homogeneous transcendental weight, but which will be cancelled later on.

We can apply analogous steps to topology  $F_4^*$ , see Fig. 3. The topology can be parametrised according to

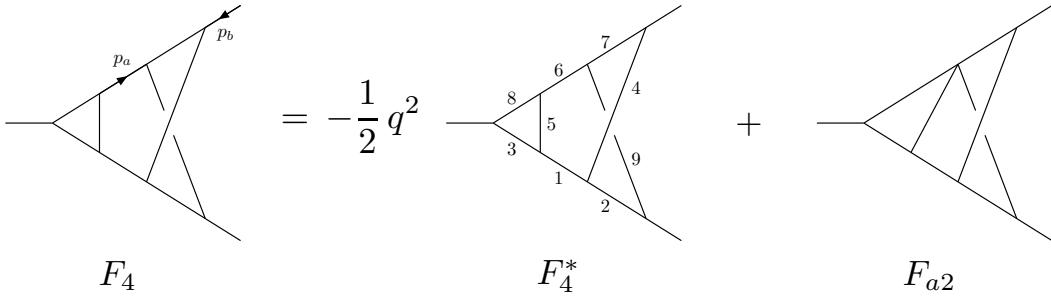
$$\{k_1, k_2, k_3, k_1 - k_2, k_1 - k_3, k_1 - q, k_1 - k_2 - p_2, k_3 - q, k_2 - p_1\} , \quad (3.4)$$

and the distribution of the momenta among the lines can be seen from Fig. 3. The integrand remains invariant under

$$\begin{aligned} k_1 &\rightarrow q - k_1 \\ k_2 &\rightarrow p_1 - k_2 \\ k_3 &\rightarrow q - k_3 , \end{aligned}$$

We now apply this transformation to the numerator  $(k_1 - p_1)^2$  of the integral  $F_4$ . This yields

$$\begin{aligned} (k_1 - p_1)^2 &\rightarrow (k_1 - p_2)^2 \\ &= k_1^2 + (k_1 - q)^2 - (k_1 - p_1)^2 - q^2 . \end{aligned} \quad (3.5)$$



**Figure 3:** Diagrammatic representation of Eq. (3.6). All symbols have the same meaning as in Fig. 2. The numbers in  $F_4^*$  indicate the position of the entries in Eq. (3.4). Diagram  $F_4$  has UT, contrary to  $F_4^*$  and  $F_{a2}$ .

We can now solve this equation for  $(k_1 - p_1)^2$  and get

$$F_4 = -\frac{1}{2} q^2 F_4^* + F_{a2} , \quad (3.6)$$

which is diagrammatically shown in Fig. 3. Again we decomposed the non-homogeneous integral  $F_4^*$  into the homogeneous integral  $F_4$  and yet another non-homogeneous auxiliary integral ( $F_{a2}$ ) which will be cancelled later on.

We can also decompose the topology  $F_5^*$ , see Fig. 4. In this case we cannot find a relation between integrals which is based on a momentum routing invariance, but a relation which is simply based on momentum conservation. The topology can be parametrised according to

$$\{k_1 - k_2, k_1 - k_3, k_1 - k_2 - k_3, k_2, k_3, k_1 - q, k_2 - q, k_1 - p_1, k_3 - p_1\} , \quad (3.7)$$

and we refer to Fig. 4 for their distributions among the lines. From momentum conservation we get

$$(k_2 - p_1)^2 = (k_1 - k_2)^2 - k_1^2 + k_2^2 + (k_1 - p_1)^2 - (k_1 - k_2 - p_1)^2 , \quad (3.8)$$

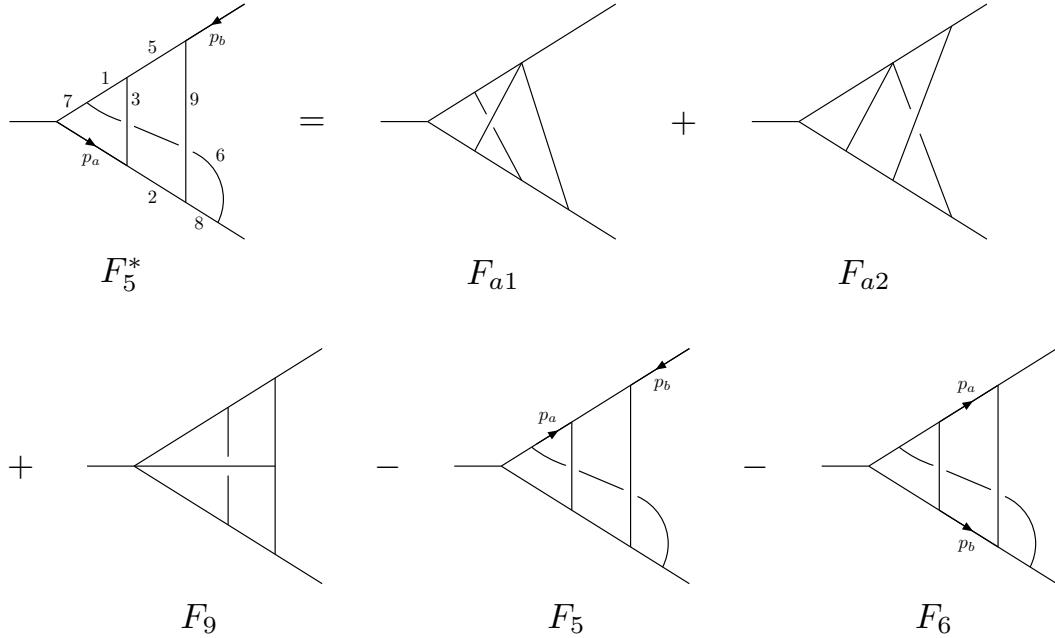
which results in

$$F_5^* = F_{a1} + F_{a2} + F_9 - F_5 - F_6 . \quad (3.9)$$

Hence we decomposed  $F_5^*$  into the homogeneous-weight diagrams  $F_5$ ,  $F_6$ , and  $F_9$ , as well as the same non-homogeneous diagrams  $F_{a1}$ , and  $F_{a2}$  which already appeared above.

We see from Eqs. (3.3), (3.6), and (3.9) that only two auxiliary diagrams of non-homogeneous weight, namely  $F_{a1}$ , and  $F_{a2}$  appear in all these relations. It turns out that the coefficients obtained from unitarity are precisely such that these integrals cancel in the expression for the form factor.

We checked all relations between integrals also at the level of their integration-by-parts (IBP) reduction [43] to master integrals using the implementation of the Laporta algorithm [44] in the REDUZE [45] code. We find that all relations obtained from momentum routing invariance in this section can actually be reproduced from solving IBP relations, which is a priori not guaranteed for a general Feynman integral topology. The  $\epsilon$ -expansions of all integrals can be found in appendix A.



**Figure 4:** Diagrammatic representation of Eq. (3.9). All symbols have the same meaning as in Fig. 2. The numbers in  $F_5^*$  indicate the position of the entries in Eq. (3.7). Diagrams  $F_5$ ,  $F_6$ , and  $F_9$  have UT, contrary to  $F_5^*$ ,  $F_{a1}$ , and  $F_{a2}$ .

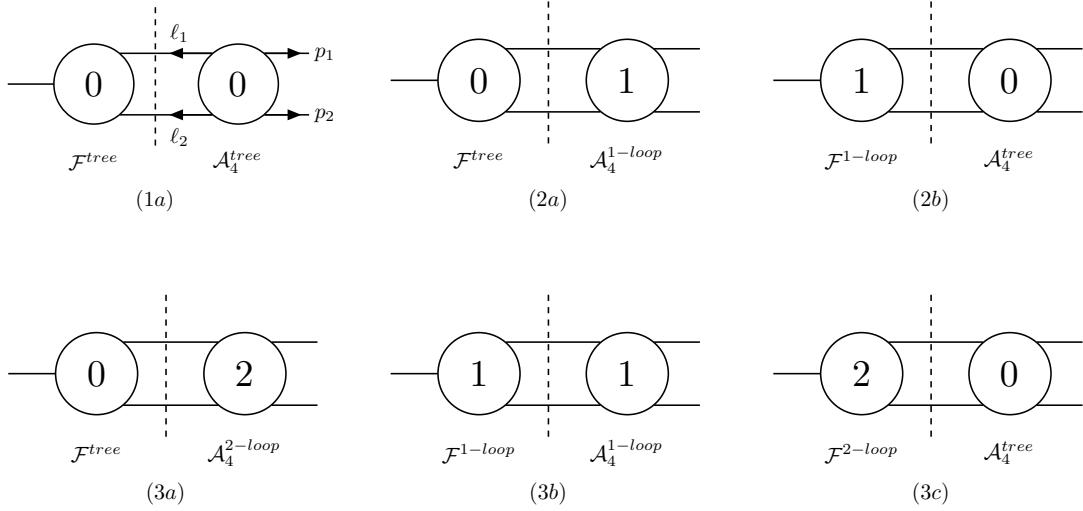
#### 4. Form factor to three loops from unitarity cuts

Here we use unitarity cuts to derive an expression for the three-loop form factor in terms of the integrals discussed in the previous section. We will compute the form factor in a perturbative expansion in the Yang-Mills coupling  $g$ , and denote the contribution at order  $g^0, g^2, g^4, g^6$  by  $\mathcal{F}_S^{tree}, \mathcal{F}_S^{1-loop}, \mathcal{F}_S^{2-loop}, \mathcal{F}_S^{3-loop}$ , respectively, and similarly for  $F_S$ . Note that this notation, convenient for the unitarity calculations, differs from the one used in Eq. (2.5).

The essential features of the unitarity-based method [46,47] that we are going to use are reviewed in the recent paper [48]. We will employ two-particle cuts, as well as generalised cuts. The two-particle cuts are very easy to evaluate, and we show an explicit example below.

In order to evaluate more complicated cuts, with many intermediate state sums to be carried out, it is extremely useful to employ a formalism that makes supersymmetry manifest. This can be done by arranging the on-shell states of  $\mathcal{N} = 4$  SYM into an on-shell supermultiplet [49]. The main advantage is that intermediate state sums appearing in the cuts become simple Grassmann integrals that can be carried out trivially [50–52]. In this way, it is easy to obtain compact analytical expressions for the cuts.

We follow the notations for unitarity cuts of ref. [53]. We start by reviewing the one- and two-loop cases as examples.



**Figure 5:** Two-particle cuts of form factors up to three loops.

#### 4.1 One-loop form factor from unitarity cuts

As a simple warmup exercise, we rederive the one-loop result from unitarity cuts, see also ref. [2]. Let us compute the two-particle cut (1a) shown in Fig. 5. It is given by

$$\mathcal{F}_S^{1-loop} \Big|_{\text{cut}(1a)} = \int \sum_{P_1, P_2} \frac{d^D k}{(2\pi)^D} \frac{i}{\ell_2^2} \mathcal{F}_S^{\text{tree}}(-\ell_1, -\ell_2) \frac{i}{\ell_1^2} \mathcal{A}_4^{\text{tree}}(\ell_2, \ell_1, p_1, p_2) \Big|_{\ell_1^2 = \ell_2^2 = 0}, \quad (4.1)$$

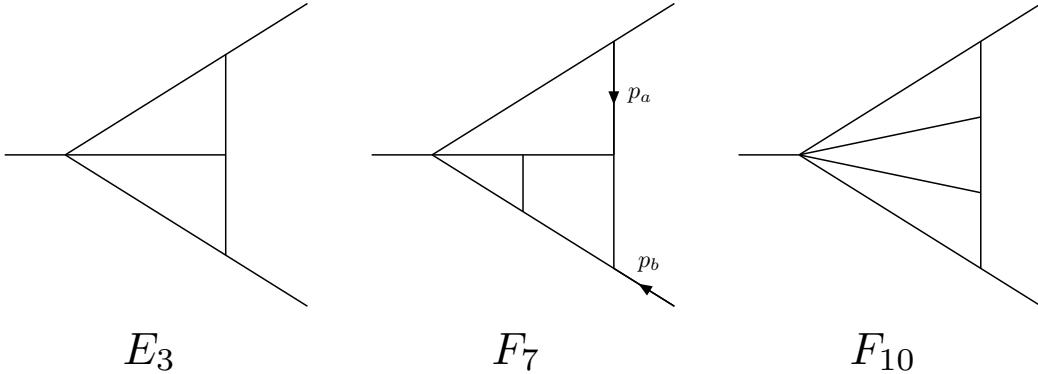
where  $\ell_1$  and  $\ell_2$  are the momenta of the cut legs, and the sum runs over all possible particles across the cut. We may use the on-shell condition  $\ell_1^2 = \ell_2^2 = 0$  in the integrand (but not on the cut propagators), since any terms proportional to such numerator factors would vanish in the cut. The four-particle tree amplitude  $\mathcal{A}_4^{\text{tree}}(\ell_2, \ell_1, p_1, p_2)$  is given in appendix C. We use the convention that all momenta are defined as outgoing.

When computing the cut of a form factor (as opposed to a colour-ordered amplitude), one has to be careful about the overall normalisation, since the possible exchange of external legs  $p_1$  and  $p_2$  leads to a factor of 2 in the cuts. When comparing cuts of the form factor to cuts of integrals, this factor cancels out. In the following we count such contributions only once.

The two-particle cuts are particularly simple to evaluate. With our choice of external states, only scalars can appear as intermediate particles, and we therefore do not need to use the spinor helicity formalism. The tree-level form factor is simply given by

$$\mathcal{F}_S^{\text{tree}}(-\ell_1, -\ell_2) = \text{Tr}(T^a T^b). \quad (4.2)$$

The necessary four-particle amplitudes are given in appendix C. The colour algebra across the cut is carried out using the  $SU(N)$  Fierz identities, see eqs. (2.10) and (2.11). It is



**Figure 6:** Diagrams that do not contribute to the form factor at two ( $E_3$ ) and three loops ( $F_7$  and  $F_{10}$ ), respectively. They have worse UV properties compared to the integrals that do appear in the form factor. The labels  $p_a$  and  $p_b$  on  $F_7$  indicate an irreducible scalar product  $(p_a + p_b)^2$  in its numerator. The other two diagrams have unit numerator.

easy to see that (4.1) becomes

$$\begin{aligned} \mathcal{F}_S^{1-loop} \Big|_{\text{cut (1a)}} &= g^2 \mu^{2\epsilon} N s_{12} \text{Tr}(T^a T^b) \int \frac{d^D k}{(2\pi)^D} \frac{i}{\ell_2^2} \frac{i}{\ell_1^2} \left( \frac{-i}{(p_1 + \ell_1)^2} + \frac{-i}{(p_2 + \ell_2)^2} \right) \Big|_{\ell_1^2 = \ell_2^2 = 0}, \\ &= -2 g^2 \mu^{2\epsilon} N s_{12} \text{Tr}(T^a T^b) \int \frac{d^D k}{i(2\pi)^D} \frac{1}{k^2 (k + p_1)^2 (k - p_2)^2} \Big|_{\text{cut (1a)}}, \end{aligned} \quad (4.3)$$

where  $s_{ij} := (p_i + p_j)^2$ , and where we have identified the cut of the one-loop form factor with the cut of the one-loop triangle integral  $D_1$ , see Fig. 1,

$$D_1 = \int \frac{d^D k}{i(2\pi)^D} \frac{1}{k^2 (k + p_1)^2 (k - p_2)^2}. \quad (4.4)$$

We can now argue that this result is exact, i.e. that we can remove the ‘‘cut (1a)’’ in Eq. (4.3). In order to do that, we have to make sure that no terms with vanishing cuts are missed. Such terms having no cuts in four dimensions can be detected in  $D$  dimensions. The two-particle cut calculation we just presented would have gone through unchanged in  $D$  dimensions, since all required amplitudes were those of scalars, and no spinor helicity identities intrinsic to four dimensions were used. A similar argument was given in ref. [53]. Therefore we conclude that in  $D$  dimensions,

$$F_S^{1-loop} = g^2 N \mu^{2\epsilon} (-q^2) 2D_1. \quad (4.5)$$

## 4.2 Two-loop form factor from unitarity cuts

We recall that at two loops, the result for the form factor is given by [1],

$$F_S^{2-loop} = g^4 N^2 \mu^{4\epsilon} (-q^2)^2 [4E_1 + E_2], \quad (4.6)$$

where the planar and non-planar ladder diagrams  $E_1$  and  $E_2$  are shown in Fig. 1.

Let us now understand this result from unitarity cuts. The unitarity cut (2b) of Fig. 5 detects the presence of the planar integral  $E_1$  only. The calculation is identical to that

of the one-loop case, with the exception that the one-loop form factor as opposed to the tree-level form factor is inserted on the l.h.s. of the cut.

The unitarity cut (2a) of Fig. 5 reveals a new feature, that was already mentioned in the introduction. On the r.h.s. of the cut we now insert the full one-loop four-point amplitude  $\mathcal{A}_4^{1-loop}$ , given explicitly in Eq. (C.2), which in addition to single trace terms also contains double trace terms. The latter would ordinarily be subleading in the expansion of powers of  $N$ , e.g. when computing a four-point amplitude at leading colour using unitarity cuts. Here, however the colour algebra gives rise to another factor of  $N$  for those terms, so that they can contribute to the form factor at the same order as the single trace terms. This explains why the non-planar integral  $E_2$  can appear in the form factor.

In principle, new terms could appear in the three-particle cut, but this is not the case. For example, the diagram  $E_3$  shown in Fig. 6 has no two-particle cuts. The absence of this diagram can be understood by the fact that it has worse UV properties compared to  $E_1$  and  $E_2$ , as we discuss in section 7. For the same reason, diagrams  $F_7$  and  $F_{10}$  from Fig. 6, the latter of which at has no three-particle cuts, will not contribute to the form factor at three loops, as we will see below.

We have also evaluated the three-particle and a generalised cut, with the result being in perfect agreement with Eq. (4.6). We found it useful to employ a manifestly supersymmetric version of the unitarity method [50]. The necessary tree-level amplitudes for the local operator of Eq. (2.1) inserted into three on-shell states were computed in refs. [2, 3]. The analytical calculation is straightforward to perform. We refrain from presenting the details since it would require introducing spinor helicity and superspace. We refer the interested reader to refs. [48, 50] for related instructive examples.

### 4.3 Three-loop form factor from unitarity cuts

We again begin by studying two-particle cuts, which are shown in the second line of Fig. 5. Again, all results for the form factors and four-point amplitudes appearing in the unitarity cuts are explicitly known, with the result for the four-point amplitudes summarized in appendix C.

When evaluating the cuts, one has a certain freedom in rewriting the answer to a given cut due to the on-shell conditions. Of course, eventually such ambiguities are fixed by the requirement that the answer must satisfy all cuts. In order to find such an expression that manifestly satisfies all cuts it is very useful to have an idea about the kind of integrals that should appear in the answer. We expect that the form factor can be expressed in terms of the integrals that have UT that were discussed in section 3. This turns out to be a very useful guiding principle.

The calculation is completely analogous to that at one and two loops. Let us start with the simplest cut (3c) from Fig. 5. It is given by

$$\mathcal{F}_S^{3-loop} \Big|_{\text{cut}(3c)} = \int \sum_{P_1, P_2} \frac{d^D k}{(2\pi)^D} \frac{i}{\ell_2^2} \mathcal{F}_S^{2-loop}(-\ell_1, -\ell_2) \frac{i}{\ell_1^2} \mathcal{A}_4^{tree}(\ell_2, \ell_1, p_1, p_2) \Big|_{\ell_1^2 = \ell_2^2 = 0}, \quad (4.7)$$

The evaluation of the cut is exactly as that considered at one loop, with the difference that we now insert the two-loop expression for the form factor into the cut, as opposed to the

tree-level one. One immediately finds

$$\mathcal{F}_S^{3-loop} \Big|_{\text{cut}(3c)} = g^6 \mu^{6\epsilon} N^3 (-q^2)^3 [8 F_1 + 2 F_3^*] \Big|_{\text{cut}(3c)}, \quad (4.8)$$

where  $F_1$  is the three-loop ladder integral shown in Fig. 7, and  $F_3^*$  is related to  $F_3$  in the same figure via the identity (3.3). In fact, we know from section 3 that  $F_3^*$  does not have uniform transcendentality. Since we do expect the final result to have this property, use Eq. (3.3) to eliminate  $F_3^*$ . When doing so, we note that the contribution of  $F_{a1}$  in that equation drops out on the cut (3c), and we have

$$F_S^{3-loop} \Big|_{\text{cut}(3c)} = g^6 \mu^{6\epsilon} N^3 (-q^2)^2 [8(-q^2) F_1 + 4 F_3 - 4 F_8] \Big|_{\text{cut}(3c)}, \quad (4.9)$$

i.e. we have succeeded in writing the two-particle cut (3c) in terms of integrals having UT only.

Similarly, one can show that the two-particle cut (3b) of Fig. 5 can be written as

$$F_S^{3-loop} \Big|_{\text{cut}(3b)} = g^6 \mu^{6\epsilon} N^3 (-q^2)^2 [8(-q^2) F_1 + 4 F_4] \Big|_{\text{cut}(3b)}. \quad (4.10)$$

This confirms the coefficient of  $F_1$ , and introduces a new integral  $F_4$ , invisible to cut (3c).

Finally, the most interesting two-particle cut is (3a), as it uses the double trace terms present in  $\mathcal{A}_4^{2-loop}$ , see appendix C. Using the identities derived in section 3, we find

$$F_S^{3-loop} \Big|_{\text{cut}(3a)} = g^6 \mu^{6\epsilon} N^3 (-q^2)^2 [8(-q^2) F_1 - 2 F_2 + 4 F_3 + 4 F_4 - 4 F_5 - 4 F_6] \Big|_{\text{cut}(3a)}. \quad (4.11)$$

Comparing equations (4.9), (4.10), and (4.11) with each other, we see that they are manifestly consistent with each other, which suggests that we are indeed working with an appropriate integral basis to describe this problem. We find that the following expression is in agreement with all two-particle cuts,

$$F_S^{3-loop} \Big|_{2\text{-part. cut}} = g^6 \mu^{6\epsilon} N^3 (-q^2)^2 [8(-q^2) F_1 - 2 F_2 + 4 F_3 + 4 F_4 - 4 F_5 - 4 F_6 - 4 F_8] \Big|_{2\text{-part. cut}}. \quad (4.12)$$

It is quite remarkable that to three loops the coefficients of all integrals are small integer numbers.

We could proceed by evaluating three- and four-particle cuts, but we find it technically simpler to study generalised cuts. To begin with, we perform a cross-check on the two-particle cut calculation above by evaluating maximal cuts where nine propagators are cut. We find perfect agreement between the two calculations. Next, we release one cut constraint to detect integrals having only eight propagators. There are several ways in which this can be done. For example, cutting all eight propagators present in integral  $F_9$  detects this integral, as well as integrals  $F_5$  and  $F_6$ . Another eight-propagator cut detects integrals  $F_2, F_5, F_6$  and  $F_7$ . The latter integral (see Fig. 6) turns out to have coefficient zero, i.e. it does not appear.

We again find perfect agreement with the contributions already known from the two-particle cuts, and find further contributions not having any two-particle cuts, like  $F_9$ . The

following expression satisfies all cuts that we have evaluated,

$$F_S^{3-loop} = g^6 \mu^{6\epsilon} N^3 (-q^2)^2 [8(-q^2) F_1 - 2 F_2 + 4 F_3 + 4 F_4 - 4 F_5 - 4 F_6 - 4 F_8 + 2 F_9]. \quad (4.13)$$

We will now argue that Eq. (4.13) is the complete result for the three-loop form factor. In fact, potential corrections to equation (4.13) can come only from seven-propagator integrals that have vanishing two-particle cuts. An example of such an integral is  $F_{10}$  shown in Fig. 6. As we will see in section 7, the appearance of such integrals is highly unlikely due to their bad UV behaviour, violating a bound based on supersymmetry power counting.

Moreover, in section 6, we will perform an even more stringent check on Eq. (4.13) by verifying the correct exponentiation of infrared divergences. In particular, this means that any potentially missing terms in equation (4.13) would have to be IR and UV finite, and vanish in all unitarity cuts that we considered.

## 5. Final result for the form factor at three loops

In the previous section we obtained the extension of Eq. (2.13) to three loops,

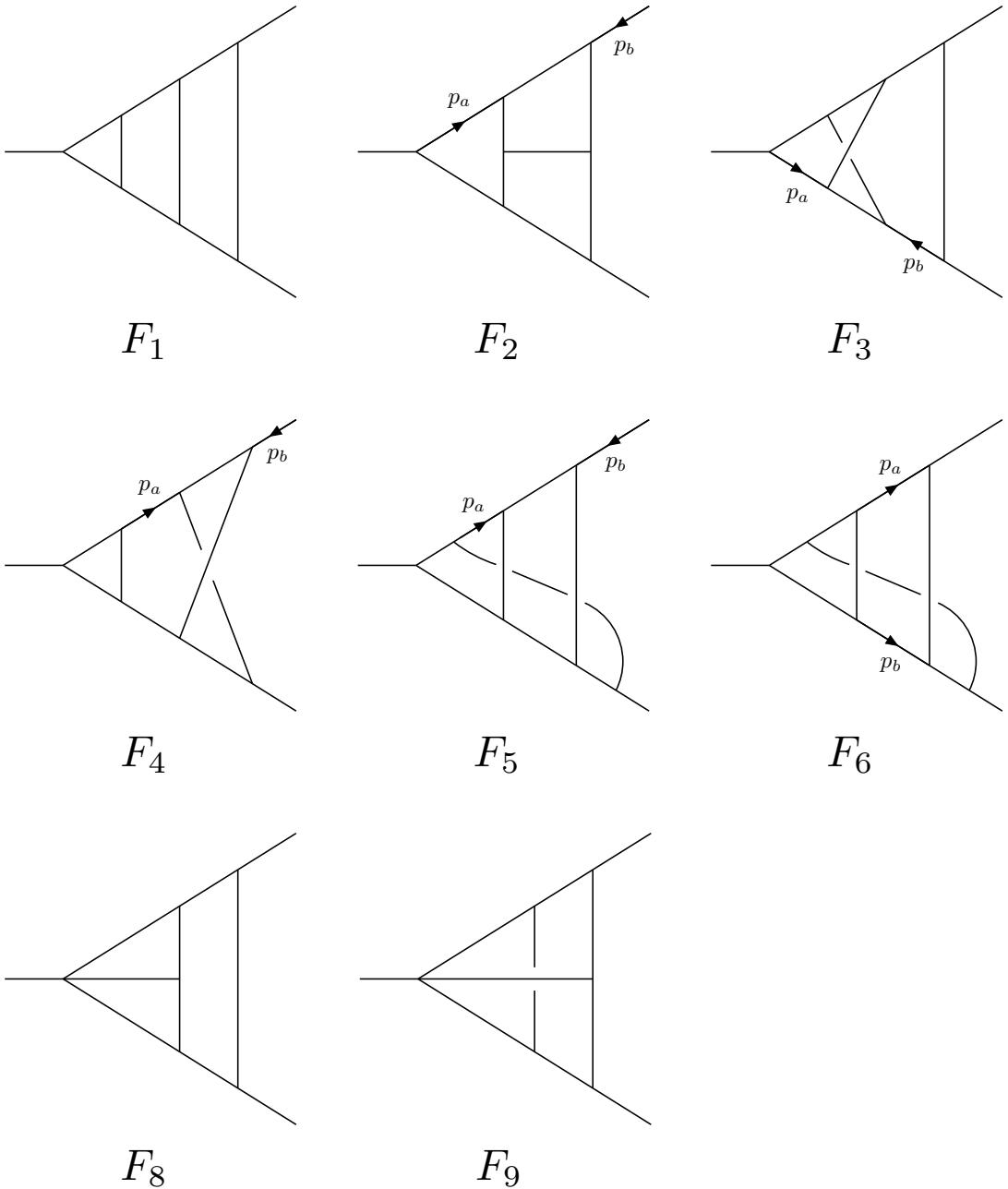
$$\begin{aligned} F_S &= 1 + g^2 N \mu^{2\epsilon} \cdot (-q^2) \cdot 2 D_1 + g^4 N^2 \mu^{4\epsilon} \cdot (-q^2)^2 \cdot [4 E_1 + E_2] \\ &\quad + g^6 N^3 \mu^{6\epsilon} \cdot (-q^2)^2 \cdot [8(-q^2) F_1 - 2 F_2 + 4 F_3 + 4 F_4 - 4 F_5 - 4 F_6 - 4 F_8 + 2 F_9] \\ &\quad + \mathcal{O}(g^8) \\ &= 1 + a x^\epsilon R_\epsilon \cdot 2 D_1^{\text{exp}} + a^2 x^{2\epsilon} R_\epsilon^2 \cdot [4 E_1^{\text{exp}} + E_2^{\text{exp}}] \\ &\quad + a^3 x^{3\epsilon} R_\epsilon^3 \cdot [8 F_1^{\text{exp}} - 2 F_2^{\text{exp}} + 4 F_3^{\text{exp}} + 4 F_4^{\text{exp}} - 4 F_5^{\text{exp}} - 4 F_6^{\text{exp}} - 4 F_8^{\text{exp}} + 2 F_9^{\text{exp}}] \\ &\quad + \mathcal{O}(a^4). \end{aligned} \quad (5.1)$$

The expressions for  $F_i$ , and  $F_i^{\text{exp}}$  are again given in appendix A. All diagrams are displayed in Fig. 7. This yields

$$\begin{aligned} F_S^{(3)} &= R_\epsilon^3 \cdot [8 F_1^{\text{exp}} - 2 F_2^{\text{exp}} + 4 F_3^{\text{exp}} + 4 F_4^{\text{exp}} - 4 F_5^{\text{exp}} - 4 F_6^{\text{exp}} - 4 F_8^{\text{exp}} + 2 F_9^{\text{exp}}] \\ &= -\frac{1}{6\epsilon^6} + \frac{11\zeta_3}{12\epsilon^3} + \frac{247\pi^4}{25920\epsilon^2} + \frac{1}{\epsilon} \left( -\frac{85\pi^2\zeta_3}{432} - \frac{439\zeta_5}{60} \right) \\ &\quad - \frac{883\zeta_3^2}{36} - \frac{22523\pi^6}{466560} + \epsilon \left( -\frac{47803\pi^4\zeta_3}{51840} + \frac{2449\pi^2\zeta_5}{432} - \frac{385579\zeta_7}{1008} \right) \\ &\quad + \epsilon^2 \left( \frac{1549}{45} \zeta_{5,3} - \frac{22499\zeta_3\zeta_5}{30} + \frac{496\pi^2\zeta_3^2}{27} - \frac{1183759981\pi^8}{7838208000} \right) + \mathcal{O}(\epsilon^3). \end{aligned} \quad (5.2)$$

We can make a very interesting observation here. For anomalous dimensions of twist two operators, there is a heuristic leading transcendentality principle [54–56], which relates the  $\mathcal{N} = 4$  SYM result to the leading transcendental part of the QCD result. We can investigate whether a similar property holds for the form factor.

For the comparison, we specify the QCD quark and gluon form factor to a supersymmetric Yang-Mills theory containing a bosonic and fermionic degree of freedom in the



**Figure 7:** Diagrams of which the three-loop form factor  $F_S^{(3)}$  in  $\mathcal{N} = 4$  SYM is built. All internal lines are massless. The incoming momentum is  $q = p_1 + p_2$ , outgoing lines are massless and on-shell, i.e.  $p_1^2 = p_2^2 = 0$ . Diagrams with labels  $p_a$  and  $p_b$  on arrow lines have an irreducible scalar product  $(p_a + p_b)^2$  in their numerator (diagrams that lack these labels have unit numerator). All diagrams displayed exhibit uniform transcendentality (UT) in their Laurent expansion in  $\epsilon = (4 - D)/2$ .

same colour representation, which is achieved by setting  $C_A = C_F = 2T_F$  and  $n_f = 1$  in the QCD result [27]. It turns out that with this adjustment the leading transcendentality

pieces of the quark and gluon form factor become equal at one, two, and three loops in all coefficients up to transcendental weight eight, i.e.  $\mathcal{O}(\epsilon^6)$ ,  $\mathcal{O}(\epsilon^4)$ , and  $\mathcal{O}(\epsilon^2)$  at one, two, and three loops, respectively. Moreover, the leading transcendentality pieces of the quark and gluon form factor coincide – up to a factor of  $2^L$  ( $L$  is the number of loops) which is due to normalisation – with the coefficients of the scalar form factor in  $\mathcal{N} = 4$  SYM computed in the present work. This again holds true at one, two, and three loops and for all coefficients up to weight eight, and serves as an important check of our result.

The question arises if the leading transcendentality principle [54–56] between QCD and  $\mathcal{N} = 4$  SYM carries over to more general quantities like scattering amplitudes, or if it is a special feature of form factors since they have only two external partons.

In fact, there are counterexamples in the case of scattering amplitudes [19]. For instance, the  $\mathcal{N} = 1$  supersymmetric one-loop four-point amplitudes [57] have a leading transcendentality piece which is not of the  $\mathcal{N} = 4$  SYM form, because it has  $1/u$  power-law factors. This makes the property we have found for the form factor even more surprising.

## 6. Logarithm of the form factor

The logarithm of the form factor is given by

$$\begin{aligned} \ln(F_S) &= \ln \left( 1 + a x^\epsilon F_S^{(1)} + a^2 x^{2\epsilon} F_S^{(2)} + a^3 x^{3\epsilon} F_S^{(3)} + \mathcal{O}(a^4) \right) \\ &= a x^\epsilon F_S^{(1)} + a^2 x^{2\epsilon} \left[ F_S^{(2)} - \frac{1}{2} \left( F_S^{(1)} \right)^2 \right] + a^3 x^{3\epsilon} \left[ F_S^{(3)} - F_S^{(1)} F_S^{(2)} + \frac{1}{3} \left( F_S^{(1)} \right)^3 \right] \\ &\quad + \mathcal{O}(a^4), \end{aligned} \quad (6.1)$$

where

$$\begin{aligned} F_S^{(1)} &= -\frac{1}{\epsilon^2} + \frac{\pi^2}{12} + \frac{7\zeta_3}{3}\epsilon + \frac{47\pi^4}{1440}\epsilon^2 + \epsilon^3 \left( \frac{31\zeta_5}{5} - \frac{7\pi^2\zeta_3}{36} \right) \\ &\quad + \epsilon^4 \left( \frac{949\pi^6}{120960} - \frac{49\zeta_3^2}{18} \right) + \epsilon^5 \left( \frac{127\zeta_7}{7} - \frac{329\pi^4\zeta_3}{4320} - \frac{31\pi^2\zeta_5}{60} \right) \\ &\quad + \epsilon^6 \left( \frac{49\pi^2\zeta_3^2}{216} - \frac{217\zeta_3\zeta_5}{15} + \frac{18593\pi^8}{9676800} \right) + \mathcal{O}(\epsilon^7), \end{aligned} \quad (6.2)$$

$$\begin{aligned} F_S^{(2)} - \frac{1}{2} \left( F_S^{(1)} \right)^2 &= \frac{\pi^2}{24\epsilon^2} + \frac{\zeta_3}{4\epsilon} + \epsilon \left( \frac{39\zeta_5}{4} - \frac{5\pi^2\zeta_3}{72} \right) + \epsilon^2 \left( \frac{235\zeta_3^2}{12} + \frac{2623\pi^6}{60480} \right) \\ &\quad + \epsilon^3 \left( \frac{73\pi^4\zeta_3}{96} - \frac{437\pi^2\zeta_5}{120} + \frac{489\zeta_7}{2} \right) \\ &\quad + \epsilon^4 \left( -66\zeta_{5,3} + \frac{1119\zeta_3\zeta_5}{10} - \frac{1351\pi^2\zeta_3^2}{216} + \frac{127\pi^8}{1296} \right) + \mathcal{O}(\epsilon^5), \end{aligned} \quad (6.3)$$

$$\begin{aligned} F_S^{(3)} - F_S^{(1)} F_S^{(2)} + \frac{1}{3} \left( F_S^{(1)} \right)^3 &= -\frac{11\pi^4}{1620\epsilon^2} + \frac{1}{\epsilon} \left( -\frac{5\pi^2\zeta_3}{54} - \frac{2\zeta_5}{3} \right) - \frac{13\zeta_3^2}{9} - \frac{193\pi^6}{25515} \\ &\quad + \epsilon \left( -\frac{107\pi^4\zeta_3}{1620} + \frac{187\pi^2\zeta_5}{108} - \frac{21181\zeta_7}{144} \right) \end{aligned}$$

$$+\epsilon^2 \left( -\frac{1421}{45} \zeta_{5,3} - \frac{1922 \zeta_3 \zeta_5}{3} + \frac{1057 \pi^2 \zeta_3^2}{108} - \frac{994807 \pi^8}{17496000} \right) \\ + \mathcal{O}(\epsilon^3) . \quad (6.4)$$

The poles of the logarithm of the form factor have the generic structure [58]

$$\ln(F_S) = \sum_{L=1}^{\infty} a^L x^{L\epsilon} \left[ -\frac{\gamma^{(L)}}{4(L\epsilon)^2} - \frac{\mathcal{G}_0^{(L)}}{2L\epsilon} \right] + \mathcal{O}(\epsilon^0) , \quad (6.5)$$

with the  $L$ -loop cusp  $\gamma^{(L)}$  and collinear  $\mathcal{G}_0^{(L)}$  anomalous dimensions [59] given by

$$\gamma(a) = \sum_{L=1}^{\infty} a^L \gamma^{(L)} = 4a - 4\zeta_2 a^2 + 22\zeta_4 a^3 + \mathcal{O}(a^4) , \quad (6.6)$$

$$\mathcal{G}_0(a) = \sum_{L=1}^{\infty} a^L \mathcal{G}_0^{(L)} = -\zeta_3 a^2 + \left( 4\zeta_5 + \frac{10}{3} \zeta_2 \zeta_3 \right) a^3 + \mathcal{O}(a^4) . \quad (6.7)$$

We observe that the vanishing of the  $\mathcal{O}(\epsilon^0)$ -term in the logarithm of the two-loop form factor [1] appears to be a coincidence, which does not reproduce at three loops. The finite part of the  $\mathcal{N} = 4$  form factor does therefore not exponentiate, as could have been conjectured from the two-loop result.

## 7. Ultraviolet divergences in higher dimensions

Scattering amplitudes and form factors in  $\mathcal{N} = 4$  super Yang-Mills are ultraviolet (UV) finite in four dimensions. It is interesting to ask in what dimension, called critical dimension  $D_c$ , they first develop UV divergences. This question is of theoretical interest in the context of the discussion of possible finiteness of  $\mathcal{N} = 8$  supergravity, see e.g. [38] and references therein. More practically, bounds on the critical UV dimension at a given loop order can also be a useful cross-check of computations, or constrain the types of loop integrals that can appear.

There is a bound on the critical dimension based on power counting for supergraphs and the background field method. The one-loop case is special due to some technical issue with ghosts, but there is a bound for  $L > 1$  loops [60, 61],

$$D_c(L) \geq 4 + \frac{2(\mathcal{N} - 1)}{L} , \quad L > 1 , \quad (7.1)$$

such that for  $D < D_c$  the theory is UV finite. The bound (7.1) depends on the number  $\mathcal{N}$  of supersymmetries that can be realized off-shell. The maximal amount of supersymmetry can be realised using an  $\mathcal{N} = 3$  harmonic superspace action for  $\mathcal{N} = 4$  super Yang-Mills [39]. Taking thus  $\mathcal{N} = 3$  in (7.1) we have

$$D_c(L) \geq 4 + \frac{4}{L} , \quad L > 1 . \quad (7.2)$$

Equation (7.1) is a lower bound for  $D_c$ , and in some cases it can be too conservative. For example, in the case of scattering amplitudes, studying and excluding potential

counterterms bounds on the critical dimension can sometimes be improved, see the reviews [62, 63]. Investigations of UV properties of four-particle scattering amplitudes have shown that their ultraviolet behaviour is better than expected [64]. Their critical dimension at two and three loops was shown to be 7 and 6, respectively, suggesting the improved bound  $D_c(L) \geq 4 + 6/L$ . The one-loop case is exceptional, but for completeness we note that  $D_c(L = 1) = 8$  for the four-particle scattering amplitude.

We can now study the UV properties for  $D > 4$  of the form factor that we have computed. There is no statement from Eq. (7.2) for the one-loop case, but one can easily see that  $D_c(L = 1) = 6$ . For the two-loop form factor, the bound (7.2) is actually saturated since the two-loop form factor develops its first ultraviolet divergence at  $D_c(L = 2) = 6$ . Moreover, it turns out that in  $D = 6 - 2\epsilon$  dimensions the leading  $1/\epsilon^2$  UV-pole is given by the leading UV-pole of the two-loop planar ladder diagram  $E_1$ , and that  $E_2$  has only a simple  $1/\epsilon$  pole.

At three loops, Eq. (7.2) becomes  $D_c \geq 16/3$ . First of all, we see by power counting that diagrams  $F_7$  and  $F_{10}$  (see Fig. 6) both have a UV divergence in  $D = 14/3$  dimensions, which would violate the supersymmetry bound (7.2). This comes close to explaining why their coefficients are zero, and why other integrals having seven or fewer propagators do not appear. A small caveat is that it may not always be possible to write the answer in a form such that the UV properties are manifest: one could have a linear combination of integrals that individually have worse UV properties than expected, but with appropriate UV behaviour of the linear combination. However, as we will see presently, we can make the UV properties of the three-loop form factor completely manifest.

At two loops we found that the bound from superspace counting was saturated. We can ask whether the same happens at three loops, i.e. do we have  $D_c(L = 3) = 16/3$ ? It turns out that the three-loop form factor is better behaved in the UV than suggested by this equation. It is finite in  $D = 16/3$  and only develops a UV divergence at  $D_c(L = 3) = 6$ . In order to see this, we take the three-loop expression (4.13) and trade  $F_3$ ,  $F_4$  and  $F_5$  for the non-UT integrals  $F_3^*$ ,  $F_4^*$  and  $F_5^*$  by means of Eqs. (3.3), (3.6), and (3.9), respectively, which leads to

$$F_S^{3-loop} \propto (-q^2) [8F_1 + 2F_3^* + 2F_4^*] - 2F_2 + 4F_5^* - 2F_9. \quad (7.3)$$

Counting numerators as propagators with negative powers, we see that the three integrals in the bracket have nine propagators each, whereas the last three integrals have only eight propagators. Since there are no sub-divergences in  $D = 16/3$  we can calculate the leading UV pole by simply giving all propagators (and also all numerators<sup>1</sup>) a common mass  $m$  and by setting the external momenta  $p_1 = p_2 = 0$ . Then the first three integrals are finite by naïve power counting, and the last three integrals become equal, and cancel due to their pre-factors. This renders the three-loop form factor finite in  $D = 16/3$  dimensions. One can see the UV finiteness of the  $\mathcal{N} = 4$  SYM form factor in  $D = 16/3$  also in another, more elegant way. We start again from Eq. (7.3), and add zero in the guise of

$$+ 2F_7^* - 2F_7^*, \quad (7.4)$$

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<sup>1</sup>Whether or not we give a mass to the numerators changes the expressions only by integrals with nine propagators each. The latter are finite in  $D = 16/3$  by naïve power counting.

where  $F_7^*$  is  $F_7$  (see Fig. 6) with unit numerator. This choice is particularly convenient since  $F_7^*$  is a subtopology of both,  $F_2$  and  $F_5^*$ . It is obtained from  $F_2$  by shrinking the line labelled  $p_a$  in Fig. 7. Alternatively,  $F_7^*$  is obtained from  $F_5^*$  by shrinking line number 7 in Fig. 4. In both cases one subsequently has to set the respective numerator to unity. Hence we can rewrite (7.3) as

$$F_S^{3-loop} \propto (-q^2) [8F_1 + 2F_3^* + 2F_4^*] - 2(F_2 - F_7^*) + 2(2F_5^* - F_7^* - F_9) . \quad (7.5)$$

If we adopt for  $F_2$  the parametrisation

$$\{k_1, k_1 + p_1, k_2, k_2 + p_2, k_3 - p_2, k_3 + p_1, k_1 + k_2, k_1 - k_3, k_2 + k_3\} , \quad (7.6)$$

and write  $(F_2 - F_7^*)$  on a common denominator, the numerator of the latter expression reads

$$k_3^2 - (k_3 - p_2)^2 \quad (7.7)$$

and hence vanishes in the aforementioned UV limit. In complete analogy, we take the parametrisation (3.7) for  $F_5^*$  and write  $(2F_5^* - F_7^* - F_9)$  on a common denominator, whose numerator becomes

$$[(k_2 - p_1)^2 - k_2^2] + [(k_2 - p_1)^2 - (k_2 - p_1 - p_2)^2] , \quad (7.8)$$

which clearly also vanishes upon taking the UV limit. Hence Eqs. (7.3) and (7.5) make the UV properties of the form factor manifest. This is very similar to how the UV properties of four-particle amplitudes can be made manifest, see e.g. ref. [64].

It is now interesting to investigate the UV properties of the form factor in  $D = 6 - 2\epsilon$  dimensions. Since the vanishing of  $(F_2 - F_7^*)$  and  $(2F_5^* - F_7^* - F_9)$  should be independent of the number of dimensions, we can simply look at the expression

$$8F_1 + 2F_3^* + 2F_4^* , \quad (7.9)$$

and the corresponding integrals at one and two loops. Introducing a common propagator mass and neglecting external momenta one finds

$$\begin{aligned} 2D_1^{\text{UV}} &\stackrel{D=6-2\epsilon}{=} S_\Gamma [m^2]^{-\epsilon} \left\{ -\frac{1}{\epsilon} - \frac{\pi^2}{6} \epsilon - \frac{7\pi^4}{360} \epsilon^3 + \mathcal{O}(\epsilon^5) \right\} , \\ 4E_1^{\text{UV}} + E_2^{\text{UV}} &\stackrel{D=6-2\epsilon}{=} S_\Gamma^2 [m^2]^{-2\epsilon} \left\{ \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} + \left[ \frac{1}{2} + \frac{\pi^2}{6} - \frac{1}{5} a_\Phi \right] + \mathcal{O}(\epsilon) \right\} , \\ 8F_1^{\text{UV}} + 2F_3^{\text{UV}} + 2F_4^{\text{UV}} &\stackrel{D=6-2\epsilon}{=} S_\Gamma^3 [m^2]^{-3\epsilon} \left\{ -\frac{1}{6\epsilon^3} - \frac{1}{2\epsilon^2} \right. \\ &\quad \left. + \frac{1}{\epsilon} \left[ \frac{\zeta_3}{3} - \frac{\pi^2}{12} - \frac{13}{9} + \frac{1}{5} a_\Phi \right] + \mathcal{O}(\epsilon^0) \right\} , \end{aligned} \quad (7.10)$$

where  $S_\Gamma$  is defined in appendix A, and (see e.g. [65])

$$a_\Phi = \Phi(-\frac{1}{3}, 2, \frac{1}{2}) + \frac{\pi \ln(3)}{\sqrt{3}} , \quad (7.11)$$

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{[(k+a)^2]^{s/2}} , \quad (7.12)$$

$$\Phi(-\frac{1}{3}, 2, \frac{1}{2}) = 4\sqrt{3} \text{Im} \left[ \text{Li}_2 \left( \frac{i}{\sqrt{3}} \right) \right] = -\frac{\pi \ln(3)}{\sqrt{3}} + \frac{10}{\sqrt{3}} \text{Cl}_2 \left( \frac{\pi}{3} \right) , \quad (7.13)$$

and  $\text{Cl}_2$  is the Clausen function. Hence we find that up to three loops the form factor at each loop-order has  $D_c = 6$ . Moreover, it turns out that for  $D = 6 - 2\epsilon$  the leading  $1/\epsilon^L$  UV-pole is at each loop order given by the leading UV-pole of the respective  $L$ -loop planar ladder diagram. Since at  $D = 6 - 2\epsilon$  there might be issues due to the presence of sub-divergences, we also computed the UV divergences using a different regulator. After having taken the soft limit, we re-insert some external momentum into the graph to serve as IR regulator, instead of the mass (essentially, one nullifies one of the  $p_i$  and takes the other one off-shell). In this way one obtains massless propagator type integrals which lead to the following result

$$\begin{aligned} 2D_1^{\text{UV}} &\stackrel{D=6-2\epsilon}{=} S_\Gamma(-q^2)^{-\epsilon} \left\{ -\frac{1}{\epsilon} - 2 - 4\epsilon + (2\zeta_3 - 8)\epsilon^2 + \mathcal{O}(\epsilon^3) \right\}, \\ 4E_1^{\text{UV}} + E_2^{\text{UV}} &\stackrel{D=6-2\epsilon}{=} S_\Gamma^2(-q^2)^{-\epsilon} \left\{ \frac{1}{2\epsilon^2} + \frac{5}{2\epsilon} + \left[ \frac{53}{6} - \zeta_3 \right] + \mathcal{O}(\epsilon) \right\}, \\ 8F_1^{\text{UV}} + 2F_3^{\text{UV}} + 2F_4^{\text{UV}} &\stackrel{D=6-2\epsilon}{=} S_\Gamma^3(-q^2)^{-3\epsilon} \left\{ -\frac{1}{6\epsilon^3} - \frac{3}{2\epsilon^2} + \frac{1}{\epsilon} \left[ \frac{4\zeta_3}{3} - \frac{79}{9} \right] + \mathcal{O}(\epsilon^0) \right\}. \end{aligned} \quad (7.14)$$

As expected, the leading  $\epsilon^{-L}$  divergence at  $L$  loops is independent of the regulator, while the subleading terms are not. However, when considering  $\log(F_S)$  in the UV limit there are only simple  $1/\epsilon$  poles up to three loops. Moreover, these poles are identical in both regularisation schemes (7.10) and (7.14), and read

$$\ln(F_S^{\text{UV}}) \stackrel{D=6-2\epsilon}{=} -\frac{\alpha}{\epsilon} + \frac{\alpha^2}{\epsilon} \frac{1}{2} + \frac{\alpha^3}{\epsilon} \left( \frac{\zeta_3}{3} - \frac{17}{18} \right) + \mathcal{O}(\alpha^4, \epsilon^0), \quad \text{with } \alpha = -q^2 \frac{g^2 N}{(4\pi)^3}. \quad (7.15)$$

Let us now discuss this result.

Despite the fact that the form factor is better behaved in the UV than expected, one may wonder why the four-particle amplitudes at one- and two loops are even better behaved in the UV than the form factor. This is due to the fact that there are specific counterterms for the local composite operator  $\mathcal{O}(x)$  in higher dimensions. Another way of saying this is in terms of operator mixing. We note that in  $D$  dimensions, the coupling constant  $g$  has dimension  $(4 - D)/2$ . Therefore, in  $D = 6$ , the operator  $\text{tr}(\phi^2)$  can mix at one loop with the operator  $g^2 \square \text{tr}(\phi^2)$ , and other operators having the same quantum numbers (we have dropped  $SU(4)$  indices for simplicity). Another reason for the better UV behaviour of the four-point amplitudes, at least in the planar limit, is the fact that amplitudes have a dual conformal symmetry, which implies that the difference between the number of propagator factors and numerator factors is four for any loop, whereas form factors are not dual conformal invariant and therefore can have fewer propagators per loop.

## 8. Discussion and conclusion

In this paper, we extended the calculation of the two-particle form factor in  $\mathcal{N} = 4$  SYM of ref. [1] to the three-loop order. We employed the unitarity-based method to obtain the answer in terms of loop integrals. The result contains both planar and non-planar integrals.

The form factor can be expressed in several ways in terms of loop integrals that make different properties manifest. One way of writing it, Eq. (4.13) is in terms of integrals all having uniform transcendentality (UT). Other forms, Eqs. (7.3) and (7.5), do not have this property, but in turn have the advantage of making the ultraviolet properties of the form factor manifest. In order to see the connection between the two representations, we derived identities between non-planar integrals based on reparametrisation invariances.

We evaluated the form factor in dimensional regularisation by reexpressing the integrals appearing in it in terms of conventionally used master integrals, c.f. Eq. (B.1), whose  $\epsilon$  expansion is known. This allowed us to evaluate the form factor to  $\mathcal{O}(\epsilon^2)$ . We verified the expected exponentiation of infrared divergences, with the correct values at three loops of the cusp and collinear anomalous dimensions.

We observed that the heuristic leading transcendentality principle that relates anomalous dimensions in QCD with those in  $\mathcal{N} = 4$  SYM holds also for the form factor. We checked this principle to three loops, up to and including terms of transcendental weight eight.

We also studied the ultraviolet (UV) properties of the form factor in higher dimensions. We found that at three loops the UV behaviour is better than suggested by a supersymmetry argument. Based on power counting one would expect three-loop integrals having 8 propagators (or nine propagators, and one loop-dependent numerator factor) to diverge in  $D = 16/3$  dimensions. However, we find that the particular linear combinations of integrals appearing in the form factor is in fact finite in this dimension, and diverges only in  $D = 6$ . We found a form, Eqs. (7.3) and (7.5), where this is manifest, and computed the leading UV divergence of  $\log(F_S)$  in  $D = 6 - 2\epsilon$  dimensions.

There are a number of interesting further directions.

It is interesting to compare the UV behaviour of the form factor to that of four-particle scattering amplitudes. While there are differences due to specific counterterms allowed for composite operators, they both share the property of having better UV behaviour than expected. It would be interesting if one could understand the UV behaviour of the form factor a priori, perhaps based on the absence of potential counterterms, or from string theory arguments.

We remark that the representations of the form factor in terms of UT integrals, Eq. (4.13), or those making its ultraviolet properties manifest, Eq. (7.5), are simpler than that in terms of conventionally used master integrals. This may indicate that, even beyond  $\mathcal{N} = 4$  SYM, there exists a basis of integrals in terms of which the result looks simpler. Similar observations about the simplicity of loop integrands and integrals in the case of planar scattering amplitudes were also made in refs. [66] and [17].

A further extension of this work could be to investigate generalised form factors with more on-shell external legs. At one-loop even all-multiplicity results could be envisaged [2–5]. At two loops, at least the three-particle form factors should be computable in a relatively straightforward manner, since the relevant integrals (two-loop four-point functions with one external leg off-shell, [67]) are known from the calculation of QCD amplitudes for the  $1 \rightarrow 3$  decay kinematics [68, 69].

The form factor studied in this paper has a very rich structure, similar to that of scattering amplitudes. Planar loop integrands of scattering amplitudes, just like tree amplitudes, satisfy powerful recursion relations [66]. It would be extremely interesting to extend the applicability of recursion relations to the non-planar case, and the form factor studied here is perhaps the simplest case of this type where non-planar integrals appear.

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## A. Explicit results of integrals

In this appendix we list explicit expressions of the integrals that appear as building blocks of the form factor. Our integration measure per loop reads

$$\int \frac{d^D k}{i(2\pi)^D}, \quad (\text{A.1})$$

and we define the pre-factor

$$S_\Gamma = \frac{1}{(4\pi)^{D/2} \Gamma(1 - \epsilon)}. \quad (\text{A.2})$$

A generic integral  $I$  can be decomposed according to

$$I = S_\Gamma^L [-q^2 - i\eta]^{n-L\epsilon} \cdot I^{\text{exp}}, \quad (\text{A.3})$$

where  $L$  is the number of loops, and the integer  $n$  is fixed by dimensional arguments.  $I^{\text{exp}}$  contains the Laurent expansion about  $\epsilon = 0$ .

We start with the one-loop integral

$$\begin{aligned} D_1 &= S_\Gamma [-q^2 - i\eta]^{-1-\epsilon} \cdot D_1^{\text{exp}}, \\ D_1^{\text{exp}} &= -\frac{\Gamma^2(-\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}. \end{aligned} \quad (\text{A.4})$$

At two loops the integrals read

$$E_1 = S_\Gamma^2 [-q^2 - i\eta]^{-2-2\epsilon} \cdot E_1^{\text{exp}},$$

$$\begin{aligned}
E_1^{\text{exp}} &= \frac{\Gamma^2(1-\epsilon)\Gamma^2(\epsilon+1)\Gamma^4(-\epsilon)}{\Gamma^2(1-2\epsilon)} - \frac{3\Gamma(1-\epsilon)\Gamma(2\epsilon+1)\Gamma^4(-\epsilon)}{2\Gamma(1-3\epsilon)} \\
&+ \frac{3\Gamma(1-2\epsilon)\Gamma(\epsilon+1)\Gamma(2\epsilon+1)\Gamma^4(-\epsilon)}{4\Gamma(1-3\epsilon)}. \tag{A.5}
\end{aligned}$$

An all-order expression for  $E_2$  can be found in [71]. The expansion in  $\epsilon$  reads

$$\begin{aligned}
E_2 &= S_\Gamma^2 [-q^2 - i\eta]^{-2-2\epsilon} \cdot E_2^{\text{exp}}, \\
E_2^{\text{exp}} &= +\frac{1}{\epsilon^4} - \frac{5\pi^2}{6\epsilon^2} - \frac{27\zeta_3}{\epsilon} - \frac{23\pi^4}{36} + \epsilon(8\pi^2\zeta_3 - 117\zeta_5) + \epsilon^2 \left( 267\zeta_3^2 - \frac{19\pi^6}{315} \right) \\
&+ \epsilon^3 \left( \frac{109\pi^4\zeta_3}{10} + 40\pi^2\zeta_5 + 6\zeta_7 \right) + \epsilon^4 \left( -264\zeta_{5,3} + 2466\zeta_3\zeta_5 - 44\pi^2\zeta_3^2 + \frac{1073\pi^8}{3024} \right) \\
&+ \mathcal{O}(\epsilon^5). \tag{A.6}
\end{aligned}$$

At three loops the integrals with uniform transcendentality (UT) are shown in Figs. 6 and 7 and read

$$\begin{aligned}
F_1 &= S_\Gamma^3 [-q^2 - i\eta]^{-3-3\epsilon} \cdot F_1^{\text{exp}}, \\
F_1^{\text{exp}} &= -\frac{1}{36\epsilon^6} - \frac{\pi^2}{12\epsilon^4} - \frac{31\zeta_3}{18\epsilon^3} - \frac{23\pi^4}{216\epsilon^2} + \frac{1}{\epsilon} \left( -\frac{5\pi^2\zeta_3}{6} - \frac{49\zeta_5}{2} \right) \\
&- \frac{43\zeta_3^2}{18} - \frac{5657\pi^6}{68040} + \epsilon \left( \frac{227\pi^4\zeta_3}{540} - \frac{7\pi^2\zeta_5}{6} - \frac{139\zeta_7}{3} \right) \\
&+ \epsilon^2 \left( -192\zeta_{5,3} + 3\zeta_3\zeta_5 + \frac{47\pi^2\zeta_3^2}{2} + \frac{959\pi^8}{12960} \right) + \mathcal{O}(\epsilon^3). \tag{A.7}
\end{aligned}$$

The integral  $F_2$  is just  $A_{9,1}^{(n)}$  from [26],

$$\begin{aligned}
F_2 &= S_\Gamma^3 [-q^2 - i\eta]^{-2-3\epsilon} \cdot F_2^{\text{exp}}, \\
F_2^{\text{exp}} &= +\frac{1}{36\epsilon^6} + \frac{\pi^2}{18\epsilon^4} + \frac{14\zeta_3}{9\epsilon^3} + \frac{47\pi^4}{405\epsilon^2} + \frac{1}{\epsilon} \left( \frac{85\pi^2\zeta_3}{27} + 20\zeta_5 \right) \\
&+ \frac{137\zeta_3^2}{3} + \frac{1160\pi^6}{5103} + \epsilon \left( \frac{829\pi^4\zeta_3}{405} + \frac{719\pi^2\zeta_5}{27} + \frac{6451\zeta_7}{9} \right) \\
&+ \epsilon^2 \left( -\frac{1184}{9}\zeta_{5,3} + 1250\zeta_3\zeta_5 - \frac{712\pi^2\zeta_3^2}{9} + \frac{593749\pi^8}{1224720} \right) + \mathcal{O}(\epsilon^3). \tag{A.8}
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
F_3 &= S_\Gamma^3 [-q^2 - i\eta]^{-2-3\epsilon} \cdot F_3^{\text{exp}}, \\
F_3^{\text{exp}} &= -\frac{1}{36\epsilon^6} + \frac{\pi^2}{9\epsilon^4} + \frac{37\zeta_3}{9\epsilon^3} + \frac{131\pi^4}{540\epsilon^2} + \frac{1}{\epsilon} \left( \frac{145\zeta_5}{3} - \frac{4\pi^2\zeta_3}{9} \right) \\
&- \frac{1352\zeta_3^2}{9} + \frac{173\pi^6}{1215} + \epsilon \left( -\frac{253\pi^4\zeta_3}{27} - \frac{62\pi^2\zeta_5}{3} - \frac{525\zeta_7}{2} \right) \\
&+ \epsilon^2 \left( \frac{6272}{5}\zeta_{5,3} - \frac{4696\zeta_3\zeta_5}{3} - \frac{712\pi^2\zeta_3^2}{9} - \frac{1301609\pi^8}{1701000} \right) + \mathcal{O}(\epsilon^3), \tag{A.9}
\end{aligned}$$

$$\begin{aligned}
F_4 &= S_\Gamma^3 [-q^2 - i\eta]^{-2-3\epsilon} \cdot F_4^{\text{exp}}, \\
F_4^{\text{exp}} &= -\frac{1}{36\epsilon^6} - \frac{\pi^2}{12\epsilon^4} - \frac{55\zeta_3}{18\epsilon^3} - \frac{11\pi^4}{216\epsilon^2} + \frac{1}{\epsilon} \left( \frac{43\pi^2\zeta_3}{6} - \frac{599\zeta_5}{6} \right) \\
&\quad - \frac{307\zeta_3^2}{18} - \frac{18797\pi^6}{68040} + \epsilon \left( -\frac{149\pi^4\zeta_3}{108} + \frac{239\pi^2\zeta_5}{2} - \frac{21253\zeta_7}{6} \right) \\
&\quad + \epsilon^2 \left( \frac{8268}{5} \zeta_{5,3} + \frac{5569\zeta_3\zeta_5}{3} - \frac{439\pi^2\zeta_3^2}{6} - \frac{184873\pi^8}{108000} \right) + \mathcal{O}(\epsilon^3), \quad (\text{A.10})
\end{aligned}$$

$$\begin{aligned}
F_5 &= S_\Gamma^3 [-q^2 - i\eta]^{-2-3\epsilon} \cdot F_5^{\text{exp}}, \\
F_5^{\text{exp}} &= +\frac{1}{12\epsilon^6} + \frac{\pi^2}{27\epsilon^4} + \frac{17\zeta_3}{9\epsilon^3} + \frac{71\pi^4}{540\epsilon^2} + \frac{1}{\epsilon} \left( \frac{71\pi^2\zeta_3}{54} + \frac{13\zeta_5}{3} \right) \\
&\quad - \frac{679\zeta_3^2}{6} + \frac{3991\pi^6}{136080} + \epsilon \left( -\frac{2837\pi^4\zeta_3}{540} + \frac{205\pi^2\zeta_5}{9} - \frac{25135\zeta_7}{24} \right) \\
&\quad + \epsilon^2 \left( \frac{4006}{3} \zeta_{5,3} - 59\zeta_3\zeta_5 - \frac{10\pi^2\zeta_3^2}{27} - \frac{14156063\pi^8}{16329600} \right) + \mathcal{O}(\epsilon^3). \quad (\text{A.11})
\end{aligned}$$

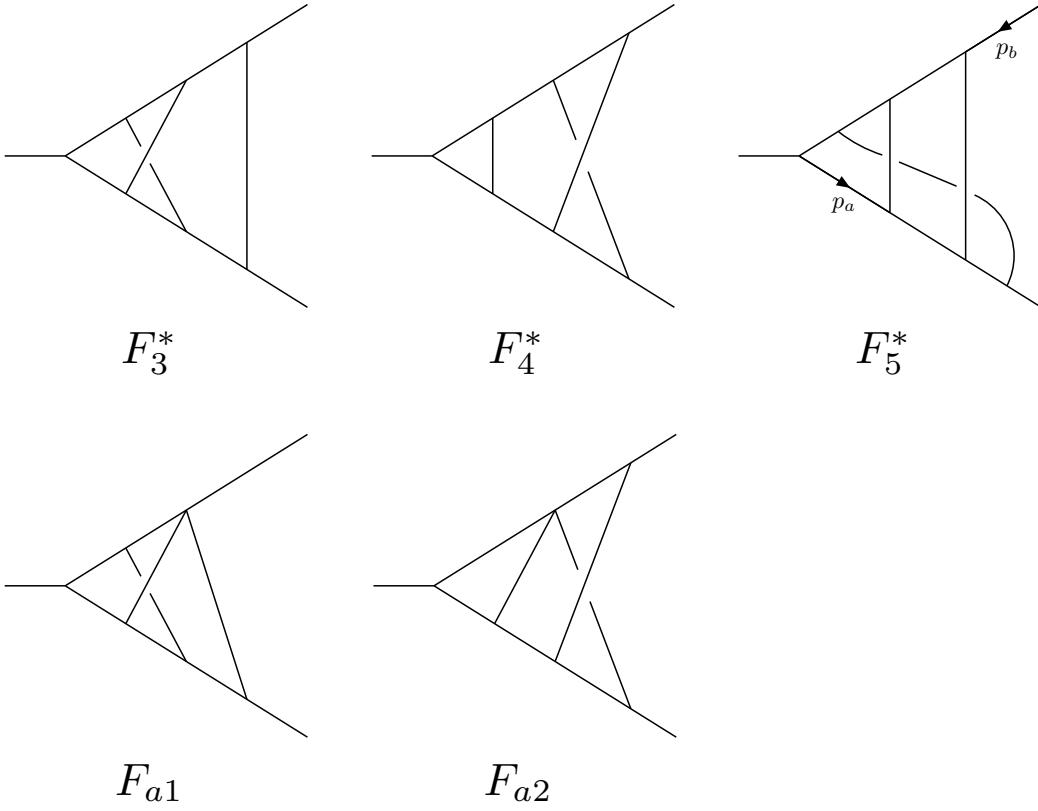
The integral  $F_6$  is just  $A_{9,2}^{(n)}$  from [26],

$$\begin{aligned}
F_6 &= S_\Gamma^3 [-q^2 - i\eta]^{-2-3\epsilon} \cdot F_6^{\text{exp}}, \\
F_6^{\text{exp}} &= +\frac{2}{9\epsilon^6} - \frac{7\pi^2}{27\epsilon^4} - \frac{91\zeta_3}{9\epsilon^3} - \frac{373\pi^4}{1080\epsilon^2} + \frac{1}{\epsilon} \left( \frac{179\pi^2\zeta_3}{27} - 167\zeta_5 \right) \\
&\quad + \frac{169\zeta_3^2}{9} - \frac{59797\pi^6}{136080} + \epsilon \left( \frac{7\pi^4\zeta_3}{30} + \frac{850\pi^2\zeta_5}{9} - \frac{18569\zeta_7}{6} \right) \\
&\quad + \epsilon^2 \left( \frac{5188}{5} \zeta_{5,3} + \frac{9362\zeta_3\zeta_5}{3} - \frac{4436\pi^2\zeta_3^2}{27} - \frac{107881603\pi^8}{81648000} \right) + \mathcal{O}(\epsilon^3). \quad (\text{A.12})
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
F_7 &= S_\Gamma^3 [-q^2 - i\eta]^{-1-3\epsilon} \cdot F_7^{\text{exp}}, \\
F_7^{\text{exp}} &= -\frac{1}{36\epsilon^6} - \frac{\pi^2}{27\epsilon^4} - \frac{7\zeta_3}{9\epsilon^3} - \frac{\pi^4}{36\epsilon^2} + \frac{1}{\epsilon} \left( \frac{20\pi^2\zeta_3}{27} - \frac{13\zeta_5}{3} \right) \\
&\quad + \frac{226\zeta_3^2}{9} - \frac{233\pi^6}{34020} + \epsilon \left( \frac{151\pi^4\zeta_3}{135} + \frac{70\pi^2\zeta_5}{9} - \frac{229\zeta_7}{6} \right) \\
&\quad + \epsilon^2 \left( \frac{248}{15} \zeta_{5,3} + \frac{1244\zeta_3\zeta_5}{3} - \frac{176\pi^2\zeta_3^2}{27} + \frac{207311\pi^8}{20412000} \right) + \mathcal{O}(\epsilon^3), \quad (\text{A.13})
\end{aligned}$$

$$\begin{aligned}
F_8 &= S_\Gamma^3 [-q^2 - i\eta]^{-2-3\epsilon} \cdot F_8^{\text{exp}}, \\
F_8^{\text{exp}} &= +\frac{1}{36\epsilon^6} + \frac{\pi^2}{27\epsilon^4} - \frac{5\zeta_3}{9\epsilon^3} + \frac{\pi^4}{108\epsilon^2} + \frac{1}{\epsilon} \left( \frac{37\zeta_5}{3} - \frac{32\pi^2\zeta_3}{27} \right) \\
&\quad + \frac{98\zeta_3^2}{9} + \frac{26\pi^6}{8505} + \epsilon \left( -\frac{4\pi^4\zeta_3}{15} - \frac{70\pi^2\zeta_5}{9} + \frac{835\zeta_7}{6} \right) \\
&\quad + \epsilon^2 \left( \frac{248}{3} \zeta_{5,3} + \frac{124\zeta_3\zeta_5}{3} + \frac{572\pi^2\zeta_3^2}{27} - \frac{16159\pi^8}{1020600} \right) + \mathcal{O}(\epsilon^3), \quad (\text{A.14})
\end{aligned}$$



**Figure 8:** Diagrams which do *not* have uniform transcendentality. As before, labels  $p_a$  and  $p_b$  on arrow lines indicate an irreducible scalar product  $(p_a + p_b)^2$  in the respective numerator, and diagrams that lack these labels have unit numerator.

$$\begin{aligned}
F_9 &= S_\Gamma^3 [-q^2 - i\eta]^{-2-3\epsilon} \cdot F_9^{\text{exp}}, \\
F_9^{\text{exp}} &= +\frac{1}{4\epsilon^6} - \frac{11\pi^2}{54\epsilon^4} - \frac{74\zeta_3}{9\epsilon^3} - \frac{43\pi^4}{180\epsilon^2} - \frac{1}{\epsilon} \left( \frac{328\zeta_5}{3} - \frac{176\pi^2\zeta_3}{27} \right) \\
&\quad + 128\zeta_3^2 - \frac{2951\pi^6}{17010} - \epsilon \left( -\frac{1021\pi^4\zeta_3}{135} - \frac{610\pi^2\zeta_5}{9} + \frac{6149\zeta_7}{6} \right) \\
&\quad - \epsilon^2 \left( \frac{392}{3}\zeta_{5,3} - \frac{11504\zeta_3\zeta_5}{3} + \frac{2876\pi^2\zeta_3^2}{27} - \frac{85171\pi^8}{1020600} \right) + \mathcal{O}(\epsilon^3), \quad (\text{A.15})
\end{aligned}$$

$$\begin{aligned}
F_{10} &= S_\Gamma^3 [-q^2 - i\eta]^{-1-3\epsilon} \cdot F_{10}^{\text{exp}}, \\
F_{10}^{\text{exp}} &= \frac{\Gamma(1-\epsilon)^2\Gamma(-\epsilon)^5\Gamma(3\epsilon)}{12\Gamma(1-4\epsilon)}. \quad (\text{A.16})
\end{aligned}$$

The integrals *without* homogeneous transcendental weight are collected in Fig. 8 and read

$$\begin{aligned}
F_3^* &= S_\Gamma^3 [-q^2 - i\eta]^{-3-3\epsilon} \cdot F_3^{*\text{exp}}, \\
F_3^{*\text{exp}} &= -\frac{1}{9\epsilon^6} + \frac{4\pi^2}{27\epsilon^4} + \frac{1}{\epsilon^3} \left( \frac{28\zeta_3}{3} + \frac{2\pi^2}{9} \right) + \frac{1}{\epsilon^2} \left( \frac{44\zeta_3}{3} - \frac{8\pi^2}{9} + \frac{7\pi^4}{15} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\epsilon} \left( -\frac{176\zeta_3}{3} + \frac{40\pi^2\zeta_3}{27} + 72\zeta_5 + \frac{32\pi^2}{9} + \frac{8\pi^4}{15} \right) - \frac{236\zeta_5}{3} - \frac{2900\zeta_3^2}{9} + \frac{56\pi^2\zeta_3}{9} \\
& + \frac{704\zeta_3}{3} + \frac{158\pi^6}{567} - \frac{32\pi^4}{15} - \frac{128\pi^2}{9} + \epsilon \left( -\frac{2816\zeta_3}{3} - \frac{224\pi^2\zeta_3}{9} - \frac{2458\pi^4\zeta_3}{135} \right. \\
& \left. - \frac{1936\zeta_3^2}{3} + \frac{944\zeta_5}{3} - \frac{232\pi^2\zeta_5}{9} - \frac{2410\zeta_7}{3} + \frac{512\pi^2}{9} + \frac{128\pi^4}{15} - \frac{262\pi^6}{945} \right) \\
& + \epsilon^2 \left( \frac{35152}{15} \zeta_{5,3} - \frac{16082\zeta_7}{3} - \frac{9640\zeta_3\zeta_5}{3} - \frac{352\pi^2\zeta_5}{3} - \frac{3776\zeta_5}{3} - \frac{5416\pi^2\zeta_3^2}{27} - \frac{512\pi^4}{15} \right. \\
& \left. + \frac{7744\zeta_3^2}{3} - \frac{224\pi^4\zeta_3}{9} + \frac{896\pi^2\zeta_3}{9} + \frac{11264\zeta_3}{3} - \frac{956008\pi^8}{637875} + \frac{1048\pi^6}{945} - \frac{2048\pi^2}{9} \right) \\
& + \mathcal{O}(\epsilon^3), \tag{A.17}
\end{aligned}$$

$$\begin{aligned}
F_4^* &= S_\Gamma^3 [-q^2 - i\eta]^{-3-3\epsilon} \cdot F_4^{*\text{exp}}, \\
F_4^{*\text{exp}} &= -\frac{1}{18\epsilon^6} + \frac{5}{18\epsilon^5} + \frac{1}{\epsilon^4} \left( -\frac{10}{9} - \frac{\pi^2}{6} \right) + \frac{1}{\epsilon^3} \left( -\frac{55\zeta_3}{9} + \frac{40}{9} - \frac{7\pi^2}{9} \right) \\
& + \frac{1}{\epsilon^2} \left( -\frac{136\zeta_3}{9} - \frac{160}{9} + \frac{28\pi^2}{9} - \frac{11\pi^4}{108} \right) + \frac{1}{\epsilon} \left( \frac{544\zeta_3}{9} + \frac{43\pi^2\zeta_3}{3} - \frac{599\zeta_5}{3} \right. \\
& \left. + \frac{640}{9} - \frac{112\pi^2}{9} - \frac{17\pi^4}{54} \right) - \frac{1108\zeta_5}{3} - \frac{307\zeta_3^2}{9} + \frac{88\pi^2\zeta_3}{9} - \frac{2176\zeta_3}{9} - \frac{18797\pi^6}{34020} \\
& + \frac{34\pi^4}{27} + \frac{448\pi^2}{9} - \frac{2560}{9} + \epsilon \left( \frac{8704\zeta_3}{9} - \frac{352\pi^2\zeta_3}{9} - \frac{149\pi^4\zeta_3}{54} - \frac{7360\zeta_3^2}{9} + \frac{4432\zeta_5}{3} \right. \\
& \left. + 239\pi^2\zeta_5 - \frac{21253\zeta_7}{3} + \frac{10240}{9} - \frac{1792\pi^2}{9} - \frac{136\pi^4}{27} - \frac{3055\pi^6}{1701} \right) + \epsilon^2 \left( \frac{16536}{5} \zeta_{5,3} \right. \\
& \left. - 17273\zeta_7 + \frac{11138\zeta_3\zeta_5}{3} + 180\pi^2\zeta_5 - \frac{17728\zeta_5}{3} - \frac{439\pi^2\zeta_3^2}{3} + \frac{29440\zeta_3^2}{9} - \frac{4846\pi^4\zeta_3}{135} \right. \\
& \left. + \frac{1408\pi^2\zeta_3}{9} - \frac{34816\zeta_3}{9} - \frac{184873\pi^8}{54000} + \frac{12220\pi^6}{1701} + \frac{544\pi^4}{27} + \frac{7168\pi^2}{9} - \frac{40960}{9} \right) \\
& + \mathcal{O}(\epsilon^3), \tag{A.18}
\end{aligned}$$

$$\begin{aligned}
F_5^* &= S_\Gamma^3 [-q^2 - i\eta]^{-2-3\epsilon} \cdot F_5^{*\text{exp}}, \\
F_5^{*\text{exp}} &= -\frac{1}{18\epsilon^6} - \frac{5}{36\epsilon^5} + \frac{1}{\epsilon^4} \left( \frac{5}{9} + \frac{\pi^2}{54} \right) + \frac{1}{\epsilon^3} \left( \frac{5\pi^2}{18} - \frac{20}{9} \right) \\
& + \frac{1}{\epsilon^2} \left( \frac{2\zeta_3}{9} + \frac{80}{9} - \frac{10\pi^2}{9} - \frac{\pi^4}{40} \right) + \frac{1}{\epsilon} \left( -\frac{8\zeta_3}{9} - \frac{77\pi^2\zeta_3}{54} + \frac{160\zeta_5}{3} \right. \\
& \left. - \frac{320}{9} + \frac{40\pi^2}{9} - \frac{59\pi^4}{540} \right) + 224\zeta_5 + \frac{4003\zeta_3^2}{18} - 8\pi^2\zeta_3 + \frac{32\zeta_3}{9} + \frac{16099\pi^6}{68040} \\
& + \frac{59\pi^4}{135} - \frac{160\pi^2}{9} + \frac{1280}{9} + \epsilon \left( -\frac{128\zeta_3}{9} + 32\pi^2\zeta_3 + \frac{151\pi^4\zeta_3}{12} + \frac{6584\zeta_3^2}{9} - 896\zeta_5 \right. \\
& \left. - \frac{445\pi^2\zeta_5}{9} + \frac{74815\zeta_7}{24} - \frac{5120}{9} + \frac{640\pi^2}{9} - \frac{236\pi^4}{135} + \frac{2519\pi^6}{2430} \right) + \epsilon^2 \left( -\frac{12518}{5} \zeta_{5,3} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{67901\zeta_7}{6} + 773\zeta_3\zeta_5 - \frac{94\pi^2\zeta_5}{3} + 3584\zeta_5 + \frac{1570\pi^2\zeta_3^2}{27} - \frac{26336\zeta_3^2}{9} + \frac{4103\pi^4\zeta_3}{135} \\
& - 128\pi^2\zeta_3 + \frac{512\zeta_3}{9} + \frac{13248257\pi^8}{5832000} - \frac{5038\pi^6}{1215} + \frac{944\pi^4}{135} - \frac{2560\pi^2}{9} + \frac{20480}{9} \Big) \\
& + \mathcal{O}(\epsilon^3), \tag{A.19}
\end{aligned}$$

$$\begin{aligned}
F_{a1} &= S_\Gamma^3 [-q^2 - i\eta]^{-2-3\epsilon} \cdot F_{a1}^{\text{exp}}, \\
F_{a1}^{\text{exp}} &= -\frac{\pi^2}{9\epsilon^3} + \frac{1}{\epsilon^2} \left( \frac{4\pi^2}{9} - \frac{22\zeta_3}{3} \right) + \frac{1}{\epsilon} \left( \frac{88\zeta_3}{3} - \frac{16\pi^2}{9} - \frac{4\pi^4}{15} \right) + \frac{118\zeta_5}{3} - \frac{28\pi^2\zeta_3}{9} \\
& - \frac{352\zeta_3}{3} + \frac{16\pi^4}{15} + \frac{64\pi^2}{9} + \epsilon \left( \frac{1408\zeta_3}{3} + \frac{112\pi^2\zeta_3}{9} + \frac{968\zeta_3^2}{3} - \frac{472\zeta_5}{3} - \frac{256\pi^2}{9} \right. \\
& \left. - \frac{64\pi^4}{15} + \frac{131\pi^6}{945} \right) + \epsilon^2 \left( -\frac{5632\zeta_3}{3} - \frac{448\pi^2\zeta_3}{9} + \frac{112\pi^4\zeta_3}{9} - \frac{3872\zeta_3^2}{3} + \frac{1888\zeta_5}{3} \right. \\
& \left. + \frac{176\pi^2\zeta_5}{3} + \frac{8041\zeta_7}{3} + \frac{1024\pi^2}{9} + \frac{256\pi^4}{15} - \frac{524\pi^6}{945} \right) + \mathcal{O}(\epsilon^3), \tag{A.20}
\end{aligned}$$

$$\begin{aligned}
F_{a2} &= S_\Gamma^3 [-q^2 - i\eta]^{-2-3\epsilon} \cdot F_{a2}^{\text{exp}}, \\
F_{a2}^{\text{exp}} &= -\frac{5}{36\epsilon^5} + \frac{5}{9\epsilon^4} + \frac{1}{\epsilon^3} \left( \frac{7\pi^2}{18} - \frac{20}{9} \right) + \frac{1}{\epsilon^2} \left( \frac{68\zeta_3}{9} + \frac{80}{9} - \frac{14\pi^2}{9} \right) + \frac{1}{\epsilon} \left( -\frac{272\zeta_3}{9} - \frac{320}{9} \right. \\
& \left. + \frac{56\pi^2}{9} + \frac{17\pi^4}{108} \right) + \frac{554\zeta_5}{3} - \frac{44\pi^2\zeta_3}{9} + \frac{1088\zeta_3}{9} - \frac{17\pi^4}{27} - \frac{224\pi^2}{9} + \frac{1280}{9} \\
& + \epsilon \left( \frac{176\pi^2\zeta_3}{9} - \frac{4352\zeta_3}{9} + \frac{3680\zeta_3^2}{9} - \frac{2216\zeta_5}{3} - \frac{5120}{9} + \frac{896\pi^2}{9} + \frac{68\pi^4}{27} + \frac{3055\pi^6}{3402} \right) \\
& + \epsilon^2 \left( \frac{17408\zeta_3}{9} - \frac{704\pi^2\zeta_3}{9} + \frac{2423\pi^4\zeta_3}{135} - \frac{14720\zeta_3^2}{9} + \frac{8864\zeta_5}{3} - 90\pi^2\zeta_5 + \frac{17273\zeta_7}{2} \right. \\
& \left. + \frac{20480}{9} - \frac{3584\pi^2}{9} - \frac{272\pi^4}{27} - \frac{6110\pi^6}{1701} \right) + \mathcal{O}(\epsilon^3). \tag{A.21}
\end{aligned}$$

At three loops we also cross-checked the major part of the integrals with the sector decomposition program FIESTA [72, 73].

## B. Form factor in terms of master integrals

Just as in QCD, the three-loop scalar form factor in  $\mathcal{N} = 4$  can be reduced to master integrals by means of the Laporta algorithm [44], for which we used the program REDUZE [45]. One obtains

$$\begin{aligned}
F_S^{(3)} &= R_\epsilon^3 \left[ + \frac{(3D-14)^2}{(D-4)(5D-22)} A_{9,1} - \frac{2(3D-14)}{5D-22} A_{9,2} - \frac{4(2D-9)(3D-14)}{(D-4)(5D-22)} A_{8,1} \right. \\
& - \frac{20(3D-13)(D-3)}{(D-4)(2D-9)} A_{7,1} - \frac{40(D-3)}{D-4} A_{7,2} + \frac{8(D-4)}{(2D-9)(5D-22)} A_{7,3} \\
& \left. - \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,4} - \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,5} \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{128(2D-7)(D-3)^2}{3(D-4)(3D-14)(5D-22)} A_{6,1} \\
& -\frac{16(2D-7)(5D-18)(52D^2-485D+1128)}{9(D-4)^2(2D-9)(5D-22)} A_{6,2} \\
& -\frac{16(2D-7)(3D-14)(3D-10)(D-3)}{(D-4)^3(5D-22)} A_{6,3} \\
& -\frac{128(2D-7)(3D-8)(91D^2-821D+1851)(D-3)^2}{3(D-4)^4(2D-9)(5D-22)} A_{5,1} \\
& -\frac{128(2D-7)(1497D^3-20423D^2+92824D-140556)(D-3)^3}{9(D-4)^4(2D-9)(3D-14)(5D-22)} A_{5,2} \\
& +\frac{4(D-3)}{D-4} B_{8,1} +\frac{64(D-3)^3}{(D-4)^3} B_{6,1} +\frac{48(3D-10)(D-3)^2}{(D-4)^3} B_{6,2} \\
& -\frac{16(3D-10)(3D-8)(144D^2-1285D+2866)(D-3)^2}{(D-4)^4(2D-9)(5D-22)} B_{5,1} \\
& +\frac{128(2D-7)(177D^2-1584D+3542)(D-3)^3}{3(D-4)^4(2D-9)(5D-22)} B_{5,2} \\
& +\frac{64(2D-5)(3D-8)(D-3)}{9(D-4)^5(2D-9)(3D-14)(5D-22)} \\
& \quad \times (2502D^5-51273D^4+419539D^3-1713688D^2+3495112D-2848104) B_{4,1} \\
& +\frac{4(D-3)}{D-4} C_{8,1} +\frac{48(3D-10)(D-3)^2}{(D-4)^3} C_{6,1} \Big] . \tag{B.1}
\end{aligned}$$

$R_\epsilon$  is given in Eq. (2.14). In order to arrive at Eq. (5.2) we have to plug in  $D = 4 - 2\epsilon$  and the  $\epsilon$ -expansions for the master integrals from Eqs. (A.7) – (A.27) of [30], together with their higher order  $\epsilon$ -terms from [32].

### C. Four-point amplitude to two loops

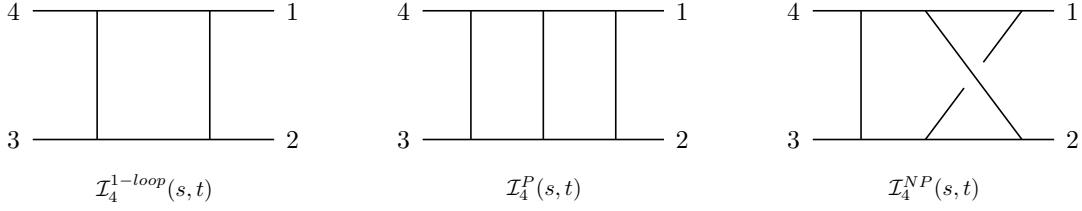
Here we summarise the known four-point amplitude in  $\mathcal{N} = 4$  super Yang-Mills to two loop order. As we have seen in the main text, both leading and subleading terms in colour are required when computing the form factor at leading colour using unitarity.

We consider four-point amplitudes in  $SU(N)$  gauge theories with all particles in the adjoint representation. Let us review the decomposition of the latter into a trace basis with partial amplitudes as coefficients [74, 75].

At tree-level, we have

$$\mathcal{A}_4^{tree} = g^2 \mu^{2\epsilon} \sum_{\sigma \in S_4/Z_4} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) A_{4;1;1}^{tree}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)), \tag{C.1}$$

where sum goes over the six non-cyclic permutations of (1234), i.e.  $S_4/Z_4 = \{(1234), (2134), (1243), (2314), (3241), (3214)\}$ . The  $A_{4;1;1}^{tree}$  are ‘partial amplitudes’. The arguments of  $\mathcal{A}$  and  $A$  in Eq. (C.1) are abbreviations, i.e. 1 stands for a given particle (gluon, fermion, or scalar) of a given helicity and momentum  $p_1^\mu$ . The  $T^a$  are the  $(N^2 - 1)$  matrices in the fundamental representation of  $SU(N)$ .



**Figure 9:** Scalar box integrals appearing in four-particle amplitudes to two loops

At loop level, double trace terms are present as well. Other possible trace terms vanish since  $\text{Tr}(T^a) = 0$  for  $SU(N)$ . We have, at one loop

$$\begin{aligned} \mathcal{A}_4^{1-loop} &= g^4 \mu^{4\epsilon} \sum_{\sigma \in S_4/Z_4} N \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) A_{4;1,1}^{1-loop}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \\ &+ g^4 \mu^{4\epsilon} \sum_{\sigma \in S_4/Z_2^3} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}}) \text{Tr}(T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) A_{4;1,3}^{1-loop}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)), \end{aligned} \quad (\text{C.2})$$

and two loops [53],

$$\begin{aligned} \mathcal{A}_4^{2-loop} &= g^6 \mu^{6\epsilon} \sum_{\sigma \in S_4/Z_4} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) \times \\ &\times \left( N^2 A_{4;1,1}^{2-loop,LC}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) + A_{4;1,1}^{2-loop,SC}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \right) \\ &+ g^6 \mu^{6\epsilon} \sum_{\sigma \in S_4/Z_2^3} N \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}}) \text{Tr}(T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) A_{4;1,3}^{2-loop}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)). \end{aligned} \quad (\text{C.3})$$

Here  $S_4/Z_2^3 = \{(1234), (1324), (1423)\}$ . The double trace terms are subleading in the expansion in powers of  $N$ . At the two-loop order, we also have the appearance of subleading-in- $N$  terms in the single trace terms, denoted by the superscript  $SC$ , while the leading-in- $N$  terms have superscript  $LC$ .

$\mathcal{N} = 4$  supersymmetric Ward identities imply that for MHV amplitudes the loop-level amplitudes are proportional to the tree-level ones, for any choice of external particles and helicities. We have

$$A_{4;1,1}^{1-loop}(1, 2, 3, 4) = -st A_{4;1,1}^{tree}(1, 2, 3, 4) \mathcal{I}_4^{1-loop}(s, t), \quad (\text{C.4})$$

where<sup>2</sup>

$$\mathcal{I}_4^{1-loop}(s, t) = \int \frac{d^D k}{i(2\pi)^D} \frac{1}{k^2 (k - p_1)^2 (k - p_1 - p_2)^2 (k + p_4)^2}, \quad (\text{C.5})$$

is the one-loop scalar box integral, see Fig. 9. The remaining subleading colour amplitudes at one loop are all equal and given by

$$A_{4;1,3}^{1-loop} = \sum_{\sigma \in S_4/Z_4} A_{4;1,1}^{1-loop}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)), \quad (\text{C.6})$$

<sup>2</sup>Note that our convention of defining loop integrals differs from that of ref. [53] by a factor of  $i$  per loop order, cf. Eq. (A.1).

which is the consequence of a  $U(1)$  decoupling identity [74].

At two loops, the partial amplitudes leading in  $N$  are given by [53]

$$A_{4;1,1}^{2-loop,LC}(1,2,3,4) = +st A_{4;1,1}^{tree}(1,2,3,4) (s\mathcal{I}_4^P(s,t) + st\mathcal{I}_4^P(t,s)) , \quad (C.7)$$

where  $\mathcal{I}_4^P(s,t)$  is the planar double box integral, see Fig. 9.

The partial amplitudes subleading in  $N$  are given by [53]

$$\begin{aligned} A_{4;1,1}^{2-loop,SC}(1,2,3,4) = & 2A_4^P(1,2;3,4) + 2A_4^P(3,4;2,1) + 2A_4^P(1,4;2,3) + 2A_4^P(2,3;4,1) \\ & - 4A_4^P(1,3;2,4) - 4A_4^P(2,4;3,1) + 2A_4^{NP}(1;2;3,4) + 2A_4^{NP}(3;4;2,1) \\ & + 2A_4^{NP}(1;4;2,3) + 2A_4^{NP}(2;3;4,1) - 4A_4^{NP}(1;3;2,4) - 4A_4^{NP}(2;4;3,1) , \end{aligned} \quad (C.8)$$

and

$$\begin{aligned} A_{4;1,3}^{2-loop}(1;2;3,4) = & 6A_4^P(1,2;3,4) + 6A_4^P(1,2;4,3) + 4A_4^{NP}(1;2;3,4) + 4A_4^{NP}(3;4;2,1) \\ & - 2A_4^{NP}(1;4;2,3) - 2A_4^{NP}(2;3;4,1) - 2A_4^{NP}(1;3;2,4) - 2A_4^{NP}(2;4;3,1) , \end{aligned} \quad (C.9)$$

where

$$A_4^P(1,2;3,4) \equiv s_{12}^2 s_{23} A_{4;1,1}^{tree}(1,2,3,4) \mathcal{I}_4^P(s_{12}, s_{23}) , \quad (C.10)$$

$$A_4^{NP}(1;2;3,4) \equiv s_{12}^2 s_{23} A_{4;1,1}^{tree}(1,2,3,4) \mathcal{I}_4^{NP}(s_{12}, s_{23}) , \quad (C.11)$$

and where  $\mathcal{I}_4^P$  and  $\mathcal{I}_4^{NP}$  are the planar and non-planar double box integral, respectively, see Fig. 9.

We remark that the expression for the double trace terms  $A_{4;1,3}^{2-loop}$  can be obtained from the single trace terms using identities derived from group theory [75, 76].

The tree-level amplitude we need has external scalars only. It is given by

$$A_{4;1,1}^{tree}(\phi_{12}(1), \phi_{12}(2), \phi_{34}(3), \phi_{34}(4)) = -i \frac{s_{12}}{s_{23}} . \quad (C.12)$$

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